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ON THE GENERATION OF WAVES BY WIND

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CONTENTS

The fully developed laminar flow of air over water confined between two infinite parallel plates was used to study nonlinear effects in the generation of surface waves. A linear stability analysis of the basic flow was made and the conditions at which small amplitude surface waves first begin to grow were determined. Then, following Stewartson & Stuart (1971), the nonlinear stability of the flow was examined and the usual parabolic equation with cubic nonlinearity obtained for the amplitude of the disturbances. The calculation of the linear stability characteristics and the coefficients appearing in the amplitude equation was a lengthy computational task, with most interest centred on the coefficient of the nonlinear terms in the amplitude equation. In two profiles, used as crude models of a boundary layer flow of air over water, the calculations indicated that, over a range of parameters, the non-linear effects would reduce the growth rate of the surface waves and hence lead to equilibrium amplitude waves.

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1. Introduction

In a recent review article, Barnett & Kenyon (1975) begin by pointing out that 'the central problem of how the wind generates waves in the ocean has not yet been solved; the primary physical mechanism(s) by which the wind makes waves has not been found'. This conclusion is essentially the same as that reached by Ursell (1956) despite the large amount of theoretical and experimental investigation into the problem which Ursell's review stimulated. The main achievements of the work in the period 1956–75 are discussed by Barnett & Kenyon (1975), and so here we will only consider how our approach to the question of wind generation of surface waves differs from previous theories.

The two rational theories of wave generation reviewed by Barnett & Kenyon are both linear theories, and hence restricted to small amplitude waves. The analysis of Phillips (1957) predicts that turbulent pressure fluctuations in the air generate small surface waves that grow linearly with time, while the work of Miles (1957), with subsequent extensions by Benjamin (1959) and Miles (1959, 1960, 1962), leads to an exponential growth rate for small waves. The primary objective of this work is to attempt to remove the restriction to small amplitude waves, and hence investigate the effect of nonlinearity on the growth of surface waves.

The analyses of Miles and Benjamin use several techniques from linear stability theory to calculate surface forces on small waves; a natural extension would be to apply the methods of nonlinear hydrodynamic stability to examine the effect of finite amplitude. However, to isolate completely nonlinear effects it is necessary to make several restrictions on the velocity profiles examined. Rather than study the nonlinear stability of a velocity profile where the water is assumed to be inviscid and the velocity in the air is given by an empirically determined turbulent mean flow profile, we shall examine the stability of laminar flow of air over water, where viscous effects are taken into account in both fluids. Calculations of the linear stability of laminar air over water profiles have been made by Lock (1954), for the case of boundary layer flow, and Feldman (1957), for semi-infinite plane Couette flow; unfortunately both calculations suffer from the same error, and the effect on the results of this mistake has not been evaluated. More recently Valenzuela (1976) has examined the growth of surface waves in turbulent mean profiles, but without decoupling the air and water motion as in the Miles (1957) calculation. All of these linear stability calculations have considered flows in infinite domains; however for definiteness, nonlinear stability calculations must be performed in finite domains, and so here we consider the flow of air and water to take place between two infinite, horizontal parallel planes. This final restriction causes the velocity profiles studied to be very different from those occurring in oceanic situations, but it is hoped that the information gained from the nonlinear stability calculation for the laminar flow profiles will be of some relevance to more general velocity profiles.

The flow studied is the fully developed laminar flow of air over water, and hence the velocity is at most a quadratic function of the distance across the channel. By allowing the upper plate to move relative to the lower boundary a range of velocity profiles can be obtained; here we examined the stability of four profiles. When the motion is driven solely by a constant pressure gradient we obtain the two-fluid analogue of plane Poiseuille flow (denoted hereafter by p.P.f.). Similarly, when the motion is due entirely to the relative motion of the two plates we obtain the profile that is the two-fluid analogue of plane Couette flow (denoted hereafter by p.C.f.). Also, a crude model of a boundary layer profile (denoted b.l.1 or b.l.2) could be obtained by adjusting the pressure gradient and velocity of the upper plate so that the velocity gradient at the upper plane is zero (figure 1). In b.l.1, p.P.f. and p.C.f. the depths of the air and water layers were equal, while in b.l.2 the water was twice as deep as the air layer.

Having chosen suitable velocity profiles, a linear stability analysis was performed to determine the neutrally stable flow conditions; this contrasts with the Miles–Benjamin approach and most experimental work, where the emphasis is on determining wave growth rates rather than the conditions at which waves first start to grow. A nonlinear stability analysis, which followed Stewartson & Stuart (1971), resulted in the amplitude equation

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}} A + \kappa A|A|^2, \qquad (1.1)$$

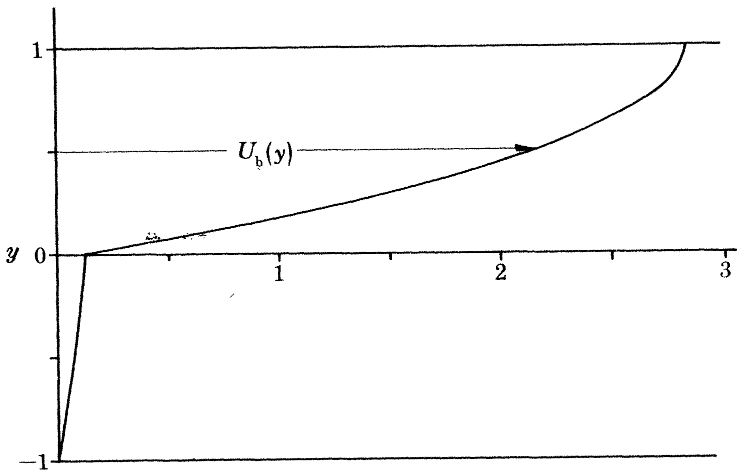

FIGURE 1. The b.l.1 velocity profile.

where the amplitude of the surface wave is proportional to A. The scaled length and time, ξ and τ, and the complex constants a_2, d_1 and κ are defined in §2.3. The constants a_2 and d_1 are properties of the flow obtained from linear theory, while the effect of nonlinear interactions is determined by κ. The main result of the numerical computations that lead to values for κ was that for the profiles b.l.1 and b.l.2 the real part of κ was, in general, negative. Thus in these profiles nonlinear effects tend to decrease the growth rate of surface waves and hence give rise to equilibrium amplitude waves.

Although (1.1) strictly only applies to laminar flow profiles, there is some experimental support for the applicability of equations similar to (1.1) to more realistic wind-wave generation situations. The experimental work of Lake & Yuen (1977) has suggested that wind-waves at fixed fetch and under steady wind conditions have properties very similar to nonlinear Stokes wavetrains. The wind-waves given by the equilibrium amplitude solutions of (1.1) (for $\kappa_r < 0$) are also similar to Stokes wavetrains, with nonlinear effects leading to changes in the phase speed of the wave and providing the sideband instability mechanism leading to the break-up of the wavetrain (Stuart & Di Prima 1978). Thus the approach used here may be relevant to the general problem of wind-wave generation, although the conclusions based on the values of κ calculated for laminar profiles may not extend to more realistic velocity distributions.

The analysis leading to (1.1) is contained in §2. The basic flow and linear stability problems are formulated in §2.1, and the conditions under which it is sufficient to consider only two-dimensional disturbances are derived. The results of the linear stability calculations are given in §2.2; the physical properties of the air and water used in the computations are given in table 1. The nonlinear stability analysis leading to the definition of the constants in (1.1) is given in §2.3 and the results of the computations to obtain numerical values for κ are presented and discussed in §3. Appendix A contains the details of the numerical schemes as well as the details of the checks on the computational procedure. Finally, appendix B contains an alternative formulation for the definition of the critical constants appearing in (1.1).

Table 1. Physical properties of air and water at 15 °C (from Batchelor 1967)

	density $\mathrm{g\,cm^{-3}}$	kinematic viscosity $\mathrm{cm^2\,s^{-1}}$
air	1.225×10^{-3}	0.145
water	0.9991	1.138×10^{-2}

Surface tension between air and water = 73.5 dyn $\mathrm{cm^{-1}}$ (73.5×10^{-3} N $\mathrm{m^{-1}}$).

2.1. The basic flow, and formulation of the linear stability problem

The basic flow considered is the steady, fully developed laminar flow of two superposed fluid layers confined between two horizontal parallel plates; the flow is generated by the combination of a pressure gradient and the movement of the upper plate (parallel to the pressure gradient) relative to the lower plate. The two fluids are taken to be immiscible and, in the steady state, the interface between the two fluids is a plane parallel to the bounding plates. We chose Cartesian coordinates $Ox^*y^*z^*$ such that, in the steady state, the plane Ox^*z^* coincides with the interface, and the y^*-axis is normal to the interface and in the opposite direction to the gravitational force. The x^*-axis is parallel to the direction of the basic flow and so, in the upper fluid, the velocity vector $\boldsymbol{U}^* = (U^*, V^*, W^*)$ has the form $(U_b^*(y^*), 0, 0)$ in the basic flow. At this point it is convenient to introduce a notational convention that will be used throughout this work, upper case symbols are used to denote the dependent variables in the upper fluid and lower case symbols denote dependent variables in the lower fluid. Thus, in the lower fluid the velocity vector is $\boldsymbol{u}^* = (u^*, v^*, w^*)$ and in the steady state $\boldsymbol{u}^*$ has the form $(u_b^*(y^*), 0, 0)$.

For the upper fluid the governing Navier–Stokes equations are

$$\left.\begin{aligned}\frac{\partial U^*}{\partial t^*}+U^*\frac{\partial U^*}{\partial x^*}+V^*\frac{\partial U^*}{\partial y^*}+W^*\frac{\partial U^*}{\partial z^*}&=-\frac{1}{\rho_2}\frac{\partial P_A^*}{\partial x^*}+\nu_2\nabla^2U^*,\\ \frac{\partial V^*}{\partial t^*}+U^*\frac{\partial V^*}{\partial x^*}+V^*\frac{\partial V^*}{\partial y^*}+W^*\frac{\partial V^*}{\partial z^*}&=-\frac{1}{\rho_2}\frac{\partial P_A^*}{\partial y^*}-g+\nu_2\nabla^2V^*\\ \frac{\partial W^*}{\partial t^*}+U^*\frac{\partial W^*}{\partial x^*}+V^*\frac{\partial W^*}{\partial y^*}+W^*\frac{\partial W^*}{\partial z^*}&=-\frac{1}{\rho_2}\frac{\partial P_A^*}{\partial z^*}+\nu_2\nabla^2W^*,\end{aligned}\right\}\tag{2.1}$$

where $\nabla^2\equiv\partial^2/\partial x^{*2}+\partial^2/\partial y^{*2}+\partial^2/\partial z^{*2}$, g is the acceleration due to gravity, and ρ_2 and ν_2 are the (constant) density and kinematic viscosity, respectively, of the upper fluid. Similarly the density and kinematic viscosity of the lower fluid are denoted by ρ_1 and ν_1 respectively.

It is convenient to remove the gravitational body force from the y^*-momentum equation by introducing a modified pressure such that

$$\left.\begin{aligned} P^*_A &= P^* - \rho_2 g y^*, \\ p^*_A &= p^* - \rho_1 g y^*. \end{aligned}\right\} \tag{2.2}$$

This transformation causes the gravitational effects to appear explicitly in the interface conditions. The boundary conditions are those of no slip on the upper and lower plates and at the interface we must have continuity of velocity and stress. Thus if the upper plate, $y^* = D_2$ has velocity U_1 relative to the lower plate, $y^* = -D_1$, and if the pressure gradient is

$$\frac{\partial p^*}{\partial x^*} = \frac{\partial P^*}{\partial x^*} = -G, \tag{2.3}$$

then the basic flow is governed by

$$\left.\begin{aligned} U^*_b &= U_1 \quad \text{on} \quad y^* = D_2, \\ 0 &= \frac{G}{\rho_2} + \nu_2 \frac{d^2 U^*_b}{dy^{*2}}, \\ u^*_b &= U^*_b, \quad \mu_1 \frac{du^*_b}{dy^*} = \mu_2 \frac{dU^*_b}{dy^*} \quad \text{on} \quad y^* = 0, \\ 0 &= \frac{G}{\rho_1} + \nu_1 \frac{d^2 u^*_b}{dy^{*2}}, \\ u_b &= 0 \quad \text{on} \quad y = -D_1. \end{aligned}\right\} \tag{2.4}$$

The solution of (2.4) is

$$U^*_b(y^*) = -\frac{GD_2^2}{2\mu_2}\left(\frac{y^*}{D_2} - 1\right)\left(\frac{y^*}{D_2} + \frac{\mu(1+d)}{d(\mu+d)}\right) + U_1 \frac{dy^*/D_2 + \mu}{\mu + d}, \tag{2.5a}$$

$$u^*_b(y^*) = -\frac{GD_2^2}{2\mu_2}\frac{\mu}{d}\left(\frac{dy^*}{D_2} + 1\right)\left(\frac{y^*}{D_2} - \frac{1+d}{\mu+d}\right) + U_1 \frac{\mu(dy^*/D_2 + 1)}{\mu + d}, \tag{2.5b}$$

where we have introduced the non-dimensional constants

$$d = D_2/D_1, \quad \mu = \frac{\rho_2 \nu_2}{\rho_1 \nu_1} = \frac{\mu_2}{\mu_1}, \quad \rho = \rho_2/\rho_1, \quad \text{and} \quad \nu = \nu_2/\nu_1. \tag{2.6}$$

The mean velocity U_m is given by

$$U_m = \frac{1}{D_1 + D_2}\left(\int_{-D_1}^{0} u^*_b \, dy^* + \int_0^{D_2} U^*_b \, dy^*\right),$$

and as the basic flow depends linearly on G and U_1, U_m may be expressed as the sum of the mean velocity caused by the pressure gradient and the mean velocity due to the motion of the upper plate. Thus if U_m is used to scale the velocity field we would have that

$$1 = U_p + U_t, \tag{2.7}$$

where U_p is the dimensionless mean velocity due to the pressure gradient and similarly U_t is the non-dimensional mean velocity due to the motion of the upper boundary. The non-dimensional basic flow is then

$$U_b(y) = \frac{U^*_b}{U_m} = -B(y-1)\left[y + \frac{\mu(1+d)}{d(\mu+d)}\right] + C(yd + \mu) \tag{2.8a}$$

and

$$u_b(y) = \frac{u^*_b}{U_m} = -\frac{\mu}{d} B(yd+1)\left(y - \frac{1+d}{\mu+d}\right) + \mu C(yd+1), \tag{2.8b}$$

where
$$y = y^*/D_2, \tag{2.8c}$$
$$B = U_p\left[\frac{\mu+d^3}{6d^2(1+d)}+\frac{1}{2}\frac{\mu(1+d)}{d(\mu+d)}\right]^{-1}, \tag{2.8d}$$
and
$$C = (1-U_p)\,\frac{2(1+d)}{\mu+d(d+2\mu)}. \tag{2.8e}$$

Thus for $U_p = 1$ the upper plate is stationary, and the flow is driven by the pressure gradient, while for $U_p = 0$ the motion is generated entirely by the relative motion of the boundaries; these two cases are the two fluid analogues of plane Poiseuille flow and plane Couette flow respectively. Another velocity profile of interest here occurs when the shear stress on the upper plate is zero. This happens when

$$U_p = \frac{2(1+d)d}{\mu+d(d+2\mu)}\left(\frac{\mu+d^3}{6d^2(1+d)}+\frac{1}{2}\frac{\mu}{d}\frac{1+d}{\mu+d}\right)\left\{1+\frac{\mu+d^3}{3d[\mu+d(d+2\mu)]}+\frac{\mu(1+d)}{d(\mu+d)}\left(1+\frac{d(1+d)}{\mu+d(d+2\mu)}\right)\right\}^{-1} \tag{2.9}$$

and it is this velocity profile which we use as a crude model of the boundary layer flow of air over water in oceanic situations.

Having defined the basic flow we can now formulate the non-dimensional equations that govern its stability. The motion is governed by the three-dimensional Navier–Stokes equations in each fluid layer, together with no-slip conditions on the boundaries and continuity of stress and velocity across the unknown interface $y = \eta(x, z, t)$. Thus, if the basic flow is perturbed by infinitesimal disturbances resulting in a velocity field

$$\boldsymbol{u} = [u_b(y), 0, 0] + \delta\{[u(y), v(y), w(y)]\, e^{i(kx+lz-kct)} + \text{c.c.}\},$$
$$\rho p = x u_b''/\mu R + \delta[\rho p(y)\, e^{i(kz+lx-kct)} + \text{c.c.}]$$
and
$$\eta = \delta[a\, e^{i(kx+lz-kct)} + \text{c.c.}],\quad |\delta| \ll 1,$$

where the velocity is scaled by U_m,

$$x = x^*/D_2,\quad z = z^*/D_2,\quad t = t^*U_m/D_2,\quad P = P^*/\rho_2 U_m^2,\quad p = p^*/\rho_2 U_m^2 \tag{2.10}$$

and c.c. denotes a complex conjugate, then the linearized equations of motion are

$$U = V = W = 0 \quad \text{on} \quad y = 1, \tag{2.11a}$$
$$ik(U_b-c)\,U + VU_b' = -ikP + R^{-1}[U'' - (k^2+l^2)\,U], \tag{2.11b}$$
$$ik(U_b-c)\,V = -P' + R^{-1}[V'' - (k^2+l^2)\,V], \tag{2.11c}$$
$$ik(U_b-c)\,W = -ilP + R^{-1}[W'' - (k^2+l^2)\,W], \tag{2.11d}$$
$$ikU + V' + ilW = 0, \tag{2.11e}$$

$$\left.\begin{aligned} &\text{velocity:} && u + au_b' = U + aU_b',\quad v = V,\quad w = W;\\ &\text{kinematic:} && ik(u_b-c)\,a = v,\quad ik(U_b-c)\,a = V;\\ &\text{tangential stress:} && u' + ikv = \mu(U' + ikV),\quad w' + ilv = \mu(W' + ilV);\\ &\text{normal stress:} && \rho p - \mathscr{F}a - \frac{2}{\nu R}v' - \mathscr{T}a(k^2+l^2) = \rho P - \rho\mathscr{F}a - \frac{2\rho}{R}V'; \end{aligned}\right\}\ \text{on}\quad y = 0 \tag{2.12}$$

$$u = v = w = 0 \quad \text{on} \quad y = -d^{-1}, \tag{2.13a}$$

$$\mathrm{i}k(u_b - c)\,u + vu_b' = -\mathrm{i}k\rho p + (\nu R)^{-1}[u'' - (k^2+l^2)\,u], \tag{2.13b}$$

$$\mathrm{i}k(u_b - c)\,v = -\rho p' + (\nu R)^{-1}[v'' - (k^2+l^2)\,v], \tag{2.13c}$$

$$\mathrm{i}k(u_b - c)\,w = -\mathrm{i}l\rho p + (\nu R)^{-1}[w'' - (k^2+l^2)\,w], \tag{2.13d}$$

$$\mathrm{i}ku + v' + \mathrm{i}lw = 0. \tag{2.13e}$$

The additional parameters are a Reynolds number, R, a Froude number, $\mathscr{F}$ and a Weber number, $\mathscr{T}$, given by

$$R = U_m D_2/\nu_2, \quad \mathscr{F} = gD_2/U_m^2, \quad \text{and} \quad \mathscr{T} = S/\rho_1 U_m^2 D_2, \tag{2.14a, b, c}$$

where S is the dimensional surface tension between the two fluids. The disturbance has wave numbers k and l in the x- and z-directions respectively and phase speed c; the flow is unstable to these disturbances when $c_i > 0$. We recall that originally, the interface conditions were to be applied on the unknown surface $y = \eta(x, z, t)$, while in (2.12) we see that the interface conditions are being applied at $y = 0$. This transformation has been obtained by noting that for small displacements of the interface, the velocity and pressure at $y = \eta$ can be adequately calculated from their respective Taylor series about $y = 0$. This device is frequently used in problems involving conditions on an unknown interface.

From (2.11), (2.12) and (2.13) we see that the stability of the basic flow is a function of seven independent non-dimensional parameters: two parameters, say ρ and ν, to describe the fluids; U_p to describe the velocity profile; a parameter d giving the geometry of the flow and three dynamic similarity parameters R, $\mathscr{F}$ and $\mathscr{T}$. However, owing to our particular interest in the combination of air over water, we can regard ρ and ν (and hence μ) as fixed. Further, the dimensional properties ρ_1, ρ_2, ν_1, ν_2 and S are fixed, as is the acceleration due to gravity, and hence we find that R, $\mathscr{F}$ and $\mathscr{T}$ are no longer independent. In fact we find that $R^{-4}\mathscr{T}^{-3}\mathscr{F}$ is a non-dimensional constant depending only on g and the physical properties of the fluids and so only two of these three quantities can be chosen to be independent parameters of the problem.

The Reynolds number is taken to be one of the independent parameters and the choice of the remaining parameter is motivated by considering how experiments on this problem would be performed. First the fluids would be chosen, and in an apparatus of fixed size the experimenter would determine the mean velocity at which the flow first became unstable. Thus in an experiment both the geometry (D_1 and D_2) and the fluid properties are fixed, and so for a given experimental situation

$$\left.\begin{aligned} \mathscr{F}R^2 &= \text{constant, say} \quad F = gD_2^3/\nu_2^2 \\ \text{and} \qquad \mathscr{T}R^2 &= \text{constant, say} \quad T = SD_2/\rho_1\nu_2^2. \end{aligned}\right\} \tag{2.15}$$

By changing the geometry of the apparatus, i.e. changing D_2, F and T are changed, even for fixed fluids. Thus the thickness of the upper layer can be regarded as a parameter relevant to the problem. However it is unsatisfactory to have a dimensional quantity as a governing parameter; this difficulty can be overcome by noting that the non-dimensional number

$$k_s = \left[\frac{(1-\rho)\,F}{T}\right]^{\frac{1}{2}} = \left[\frac{(\rho_1-\rho_2)\,g}{S}\right]^{\frac{1}{2}} D_2 \tag{2.16}$$

is proportional to D_2, the constant of proportionality being a property of the two fluids only. Thus, for fixed fluid properties, k_s reflects changes in geometry of the flow. In fact k_s is just the non-dimensional wavenumber of the inviscid surface wave with minimum phase speed.

Further justification for this choice of scaling can be obtained by considering the relation between the stability of two- and three-dimensional disturbances. Following Squire we introduce new variables given by

$$\left.\begin{aligned} m^2 &= k^2+l^2, \\ \hat{U} &= m^{-1}(kU+lW), & \hat{u} &= m^{-1}(ku+lw), \\ \hat{V} &= V, & \hat{v} &= v \\ \hat{c} &= c, & \hat{R} &= kR/m, \\ \hat{p} &= (m/k)\,p, & \hat{P} &= (m/k)\,P \\ \hat{a} &= (k/m)\,a. \end{aligned}\right\} \tag{2.17}$$

After some manipulation we obtain the system

$$\hat{U} = \hat{V} = 0 \text{ on } y=1,$$

$$\mathrm{i}m(U_b-\hat{c})\,\hat{U}+\hat{V}U_b' = -\mathrm{i}m\hat{P}+\hat{R}^{-1}(\hat{U}''-m^2\hat{U}),$$

$$\mathrm{i}m(U_b-\hat{c})\,\hat{V} = -\hat{P}'+\hat{R}^{-1}(\hat{V}''-m^2\hat{V}),$$

$$\mathrm{i}m\hat{U}+\hat{V}' = 0,$$

$$\left.\begin{aligned} \hat{u}+\hat{a}u_b' &= \hat{U}+\hat{a}U_b', \\ \hat{v} &= \hat{V}, \\ \mathrm{i}m(u_b-\hat{c})\,\hat{a} = \hat{v}, \quad \mathrm{i}m&(U_b-\hat{c})\,\hat{a} = \hat{V}, \\ \hat{u}'+\mathrm{i}m\hat{v} &= \mu(\hat{U}'+\mathrm{i}m\hat{V}), \\ \rho\hat{p}-\frac{F}{\hat{R}^2}\,\hat{a}-\frac{2}{\nu\hat{R}}\,\hat{v}'-\frac{T}{\hat{R}^2}\,m^2\hat{a} &= \rho\hat{P}-\rho\frac{F}{\hat{R}^2}\,\hat{a}-\frac{2\rho}{\hat{R}}\,\hat{V}', \end{aligned}\right\} \text{ on } y=0$$

$$\mathrm{i}m(u_b-\hat{c})\,\hat{u}+\hat{v}u_b' = -\mathrm{i}m\rho\hat{p}+(\nu\hat{R})^{-1}\,(\hat{u}''-m^2\hat{u}),$$

$$\mathrm{i}m(u_b-\hat{c})\,\hat{v} = -\rho\hat{p}'+(\nu\hat{R})^{-1}\,(\hat{v}''-m^2\hat{v}),$$

$$\mathrm{i}m\hat{u}+\hat{v}' = 0,$$

$$\hat{u} = \hat{v} = 0 \quad \text{on} \quad y=-d^{-1}.$$

Thus we see that the gravitational and surface tension parameters F and T are invariant under the Squire transformation, so that a three-dimensional disturbance with wavenumbers k and l and parameters R, F and T is equivalent to a two-dimensional disturbance with wavenumber $(k^2+l^2)^{\frac{1}{2}}$ and parameters $Rk(k^2+l^2)^{-\frac{1}{2}}$, F and T. Hence, for fixed F and T, the minimum critical Reynolds number occurs for a two-dimensional disturbance.

As we need only consider two-dimensional disturbances l, $W(y)$ and $w(y)$ can be set to zero in (2.11–13) and the remaining velocity components obtained from the stream functions $\Phi(y)$ and $\phi(y)$ by

$$\left.\begin{aligned} U &= \Phi', & u &= \phi', \\ V &= -\mathrm{i}k\Phi, & v &= -\mathrm{i}k\phi. \end{aligned}\right\} \tag{2.18}$$

Thus the system governing the stability of the basic flow is

$$\left.\begin{aligned}
&\Phi = \Phi' = 0 \quad \text{on} \quad y = 1,\\
&\mathrm{i}k(U_b - c)\,(\Phi'' - k^2\Phi) - \mathrm{i}kU_b''\Phi = R^{-1}(\Phi^{(\mathrm{iv})} - 2k^2\Phi'' + k^4\Phi),\\
&\left.\begin{aligned}
&\phi = \Phi,\\
&\phi' - \frac{u_b'\phi}{\sigma} = \Phi' - \frac{U_b'\Phi}{\sigma},\\
&\phi'' + k^2\phi = \mu(\Phi'' + k^2\Phi)\\
&\frac{1}{\nu R}(\phi''' - 3k^2\phi') + \mathrm{i}k(\phi u_b' - \phi'\sigma) + \frac{\mathrm{i}k\phi}{\sigma}\frac{(F + k^2T)}{R^2}\\
&\quad = \frac{\rho}{R}(\Phi''' - 3k^2\Phi') + \rho\mathrm{i}k(\Phi U_b' - \Phi'\sigma) + \frac{\rho\mathrm{i}k\Phi}{\sigma}\frac{F}{R^2},
\end{aligned}\right]\quad \text{on} \quad y = 0\\
&\mathrm{i}k(u_b - c)\,(\phi'' - k^2\phi) - \mathrm{i}ku_b''\phi = (\nu R)^{-1}\,(\phi^{(\mathrm{iv})} - 2k^2\phi'' + k^4\phi),\\
&\phi = \phi' = 0 \quad \text{on} \quad y = -d^{-1}
\end{aligned}\right\} \tag{2.19}$$

where $\sigma = u_b(0) - c = U_b(0) - c$.

Systems equivalent to (2.19) have been derived by Yih (1967) and more recently by Valenzuela (1976), although the relevance of F and T to the stability of two-dimensional waves was not noted by them. Earlier, Lock (1954) and Feldman (1957) had derived similar coupled Orr–Sommerfeld problems for the stability of superposed fluid layers, but unfortunately they incorrectly omitted the $u_b'\phi/\sigma$ and $U_b'\Phi/\sigma$ terms from the second interface condition, shown in (2.19).

In general the solution of (2.19), and the location of the neutral stability curve in the R, k-plane, is a computational task, and full details of the numerical techniques used are given in appendix A. Section 2.2 summarizes the computational work and discusses the results obtained, while the following two subsections contain two elementary analytical calculations, for limiting cases of (2.19), which were used to check the numerical work.

2.1.1. *The long-wave limit*

Here we consider the stability of the basic flow to long wave length disturbances. This was first done by Yih (1967), but only results for the case $\rho = 1$ were given. Our analysis parallels that given by Yih, and so we look for solutions of the form

$$\left.\begin{aligned}
c &\sim c_0 + kc_1 + \dots,\\
\phi &\sim \phi_0 + k\phi_1 + \dots,\\
\Phi &\sim \Phi_0 + k\Phi_1 + \dots,
\end{aligned}\right\}\quad \begin{aligned}&\text{as} \quad k \to 0,\\ &R,\ F,\ T \text{ fixed.}\end{aligned} \tag{2.20}$$

At $O(k^0)$ we obtain the homogeneous system

$$\left.\begin{aligned}
&\Phi_0 = \Phi_0' = 0 \quad \text{on} \quad y = 1,\\
&0 = R^{-1}\Phi_0^{(\mathrm{iv})},\\
&\left.\begin{aligned}
&\phi_0 = \Phi_0, \quad \phi_0' - \frac{u_b'\phi_0}{\sigma} = \Phi_0' - \frac{U_b'\Phi_0}{\sigma}, \quad \sigma = u_b - c_0,\\
&\phi_0'' = \mu\Phi_0'', \quad (\nu R)^{-1}\,\phi_0''' = \rho R^{-1}\Phi_0''',
\end{aligned}\right]\quad \text{on} \quad y = 0\\
&0 = (\nu R)^{-1}\,\phi_0^{(\mathrm{iv})}, \quad \phi_0 = \phi_0' = 0 \quad \text{on} \quad y = -d^{-1},
\end{aligned}\right\} \tag{2.21}$$

which has the eigen solution

$$\Phi_0 = (y-1)^2\left(y\,\frac{d}{2\mu}\,\frac{\mu-d^2}{1+d}+1\right),$$

$$\phi_0 = (y+d^{-1})^2\left(y\,\frac{d}{2}\,\frac{\mu-d^2}{1+d}+d^2\right).$$

The eigenvalue c_0 is

$$c_0 = u_b(0)+[2\mu d(1+d)\,(U_b'-u_b')]/[4\mu d(1+d)^2+(\mu-d^2)^2], \quad \text{provided} \quad \mu \neq 1. \tag{2.22}$$

The $O(k)$ system is

$$\left.\begin{aligned}
&\Phi_1 = \Phi_1' = 0 \quad \text{on} \quad y = 1,\\
&\mathrm{i}(U_b-c_0)\,\Phi_0''-\mathrm{i}U_b''\Phi_0 = R^{-1}\Phi_1^{(\mathrm{iv})},\\
&\left.\begin{aligned}
&\phi_1 = \Phi_1,\\
&\phi_1'-\frac{u_b'}{\sigma}\left(\phi_1+\frac{c_1}{\sigma}\,\phi_0\right) = \Phi_1'-\frac{U_b'}{\sigma}\left(\Phi_1+\frac{c_1}{\sigma}\,\Phi_0\right),\\
&\phi_1'' = \mu\Phi_1'',\\
&(\nu R)^{-1}\,\phi_1'''-\mathrm{i}(\phi_0'\sigma-\phi_0 u_b')+\frac{\mathrm{i}\phi_0}{\sigma}\,\frac{F}{R^2} = \rho R^{-1}\Phi_1'''-\rho\mathrm{i}(\Phi_0'\sigma-\Phi_0 U_b')+\frac{\rho\mathrm{i}\Phi_0 F}{\sigma R^2},
\end{aligned}\right]\quad \text{on} \quad y = 0\\
&\mathrm{i}(u_b-c_0)\,\phi_0''-\mathrm{i}u_b''\phi_0 = (\nu R)^{-1}\,\phi_1^{(\mathrm{iv})},\\
&\phi_1 = \phi_1' = 0 \quad \text{on} \quad y = -d^{-1},
\end{aligned}\right\} \tag{2.23}$$

and as ϕ_0, Φ_0 and c_0 are real, we see that c_1 is purely imaginary. The homogeneous part of (2.23) is the same as (2.21), so the non-homogeneous system can have a solution only if the forcing terms satisfy certain conditions. These conditions can be found by explicitly solving (2.23) (see Yih (1967)), but it is easier to introduce the adjoint functions Ψ and ψ and invoke a solvability criterion. The system adjoint to (2.21) is

$$\left.\begin{aligned}
&\Psi = \Psi' = 0 \quad \text{on} \quad y = 1,\\
&0 = R^{-1}\Psi^{(\mathrm{iv})},\\
&\left.\begin{aligned}
&\psi = \Psi, \quad \psi' = \Psi',\\
&\psi'' = \mu\Psi'', \quad \psi'''-\frac{u_b'}{\sigma}\,\psi' = \mu\left(\Psi'''-\frac{U_b'}{\sigma}\,\Psi'\right),
\end{aligned}\right]\quad \text{on} \quad y = 0\\
&0 = (\nu R)^{-1}\,\psi^{(\mathrm{iv})},\\
&\psi = \psi' \to 0 \quad \text{on} \quad y = -d^{-1},
\end{aligned}\right\} \tag{2.24}$$

with adjoint condition

$$\int_{-d^{-1}}^{0}\psi(\nu R)^{-1}\,\phi_0^{(\mathrm{iv})}\,\mathrm{d}y+\rho\int_0^1\Psi R^{-1}\Phi_0^{(\mathrm{iv})}\,\mathrm{d}y = \int_{-d^{-1}}^{0}\phi(\nu R)^{-1}\,\psi^{(\mathrm{iv})}\,\mathrm{d}y+\rho\int_0^1\Phi R^{-1}\Psi^{(\mathrm{iv})}\,\mathrm{d}y = 0. \tag{2.25}$$

(The adjoint system for the complete coupled Orr–Sommerfeld problem (2.19) is used in the nonlinear stability calculation and is derived in § 2.3.1. The above system is readily obtained from (2.35) by taking limits analogous to (2.20).)

The eigenvalue c_0 in the adjoint problem is the same as in (2.21), while the adjoint functions are

$$\Psi = (y-1)^2\left[y\,\frac{\mu-d^2+4d(1+d)}{2(\mu+d)}+1\right]$$

and

$$\psi = (y+d^{-1})^2\left[y\,\frac{d^2(\mu-d^2)-4\mu d^2(1+d)}{2(\mu+d)}+d^2\right]. \tag{2.26}$$

When the solvability condition is applied to (2.23) we obtain the following expression for c_1:

$$\begin{aligned}\frac{c_1\phi_0}{\sigma^2}(\nu R)^{-1}\,\psi''(u_b'-U_b') = &-\mathrm{i}\psi\left[(1-\rho)\,(\phi_0'\sigma-\phi_0 u_b')-\frac{\phi_0 F(1-\rho)}{\sigma R^2}\right]\\ &+\mathrm{i}\int_{-d^{-1}}^{0}\psi[(u_b-c_0)\,\phi_0''-u_b''\phi_0]\,\mathrm{d}y\\ &+\rho\mathrm{i}\int_0^1\Psi[(U_b-c_0)\,\Phi_0''-U_b''\Phi_0]\,\mathrm{d}y,\end{aligned}$$

where any function that is not integrated is evaluated at $y = 0$. To this order in k, the condition for neutral stability is that $c_1 = 0$, which gives the value

$$\begin{aligned}\frac{F}{R^2} = \sigma(\phi_0'\sigma-u_b')-\frac{\sigma}{1-\rho}\left\{\int_{-d^{-1}}^{0}\psi[(u_b-c_0)\,\phi_0''-u_b''\phi_0]\,\mathrm{d}y\right.\\ \left.+\rho\int_0^1\Psi[(U_b-c_0)\,\Phi_0''-U_b''\Phi_0]\,\mathrm{d}y\right\}\end{aligned} \tag{2.27}$$

for neutral stability. Here (2.27) has been simplified by using the fact that ϕ_0 and ψ have been normalized to unity at $y = 0$.

2.1.2. *The limit of slow basic flow*

When the Reynolds number for the basic flow becomes small we look for solutions of the form

$$\left.\begin{aligned}c &\sim R^{-1}c_0+c_1+\dots,\\ \phi &\sim \phi_0+R\phi_1+\dots,\\ \Phi &\sim \Phi_0+R\Phi_1+\dots,\end{aligned}\right\}\quad \text{as}\quad R\to 0,\quad F \text{ and } T \text{ fixed}. \tag{2.28}$$

The governing equations are then

$$\left.\begin{aligned}\Phi_0 = \Phi_0' = 0 \quad &\text{on}\quad y = 1,\\ -\mathrm{i}kc_0(\Phi_0''-k^2\Phi_0) &= \Phi_0^{(\mathrm{iv})}-2k^2\Phi_0''+k^4\Phi_0,\\ \left.\begin{aligned}\phi_0 = \Phi_0,\quad \phi_0' &= \Phi_0',\\ \phi_0''+k^2\phi_0 &= \mu(\Phi_0''+k^2\Phi_0),\\ \nu^{-1}(\phi_0'''-3k^2\phi_0')+\mathrm{i}kc_0\phi_0'&-(\mathrm{i}k\phi_0/c_0)\,(F+k^2T)\\ &= \rho(\Phi_0'''-3k^2\Phi_0')+\rho\mathrm{i}kc_0\Phi_0'-(\rho\mathrm{i}k\Phi_0/c_0)\,F,\end{aligned}\right] &\quad\text{on}\quad y = 0\\ -\mathrm{i}kc_0(\phi_0''-k^2\phi_0) &= \nu^{-1}(\phi_0^{(\mathrm{iv})}-2k^2\phi_0''+k^4\phi_0),\\ \phi_0 = \phi_0' = 0 \quad &\text{on}\quad y = -d^{-1}.\end{aligned}\right\} \tag{2.29}$$

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The equations (2.29) are recognized as those governing small disturbances in two motionless fluids, and so we can expect all perturbations to be damped. The solution of (2.29) is

$$\Phi_0 = A[\cosh k(y-1) - \cosh \lambda(y-1)] + B[\sinh \lambda(y-1) - (\lambda/k) \sinh k(y-1)],$$

where
$$\lambda^2 = k^2 - \mathrm{i}c_0 k,$$
and
$$\phi_0 = a[\cosh k(y+d^{-1}) - \cosh \gamma(y+d^{-1})] + b[\sinh \gamma(y+d^{-1}) - (\gamma/k) \sinh k(y+d^{-1})],$$

where
$$\gamma^2 = k^2 - \mathrm{i}\nu c_0 k.$$

The eigenvalue c_0 and constants A, B, a and b are obtained by enforcing the interface conditions. This leads to a homogeneous 4×4 set of equations which have non-trivial solutions only at certain values of c_0; the determination of these eigenvalues is in general a computational task. However, for air over water, where $\rho \approx 10^{-3}$ and F and T are relatively large (approximately 10^5 for $k_s = 3.65$) it can be shown that the eigenvalues can be put into two classes: there are two complex values for c_0 and an infinite sequence of purely imaginary eigenvalues. The complex eigenvalues correspond to weakly damped surface waves, one travelling in the $+x$-direction, the other in the $-x$-direction. The negative imaginary eigenvalues can be further subdivided into those associated with diffusive effects in the water (Lamb 1932, p. 628) and those due to diffusive effects in the air.

The eigenfunction associated with the latter values of c_0 have $\Phi_0 \approx O(1)$ and $\phi_0 \approx O(\rho)$, and hence represent a motion confined mainly to the air.

2.2. Results of the linear stability calculations

The coupled Orr–Sommerfeld problem (2.19) was solved numerically by using a finite difference approximation to the equations and boundary conditions. The difference scheme used was similar to that described by Osborne (1967), where the truncation error of the difference approximation is $O(h^4)$ for a uniform grid with step length h. For given R and k the eigenvalue c was found by using a Muller iteration scheme. Then for fixed k, Muller's method was used to determine the value of R_n for neutral stability and finally a Muller iteration method was used to locate the critical conditions k_c and R_c. Most of the neutral curves were determined on relatively coarse meshes, giving only 5–10% accuracy. However, when calculating the critical conditions the step size was progressively reduced until five significant figures could be guaranteed in R_c. Under these conditions k_c would usually contain at least four significant figures.

The numerical scheme developed to solve (2.19) was checked by comparing results from the program with the analytical solutions given in §§2.1.1 and 2.1.2 and with published results for problems that are special cases of (2.19). Full details of these tests and the details of the finite difference scheme are given in appendix A.

As well as providing a check on the program, the solution given in §2.1.1 also provides a starting point for the determination of the neutral curves. From §2.1.1 we see that in the limit $k \to 0$ the growth rate of disturbances is $O(k)$, and hence the Reynolds number axis is part of the neutral curve. For a given velocity profile, the Reynolds number at which the neutral curve branches off the axis is given by (2.27); the remainder of the neutral curve is then found by gradually increasing k and using the last known R_n as the initial approximation for the

Muller interation to find the current position of the neutral curve. In general this technique worked well, and some neutral curves typical of the results obtained are shown in figures 2–5.

From figure 2 we see that for the p.P.f. profile the neutral curve is a continuous curve joining the R-axis at the point given by (2.27). However, for the b.l.1 and b.l.2 profiles, the complete set of neutrally stable disturbances could not always be found by following the neutral curve away from the long-wave instability. For $k_s = 3.65$ (figure 4) the neutral curve is similar to

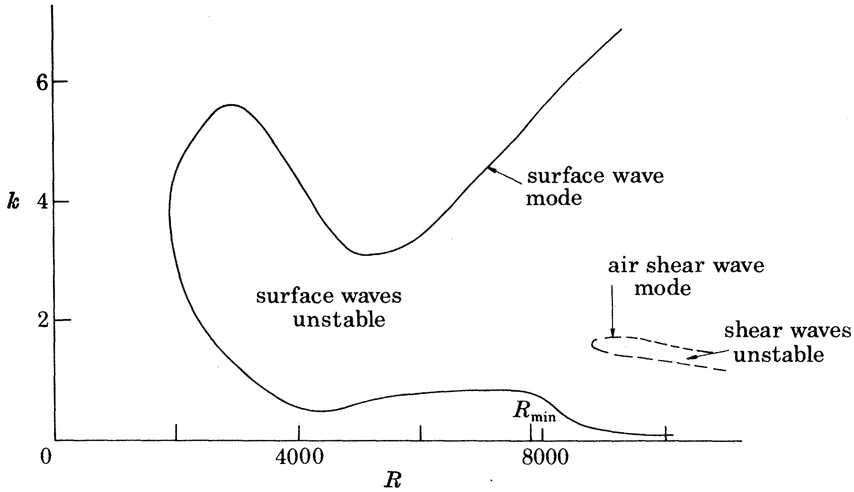

FIGURE 2. Neutral curves for p.P.f. profile with $k_s = 29.2$. R_{min}, minimum value of R for long-wave instability.

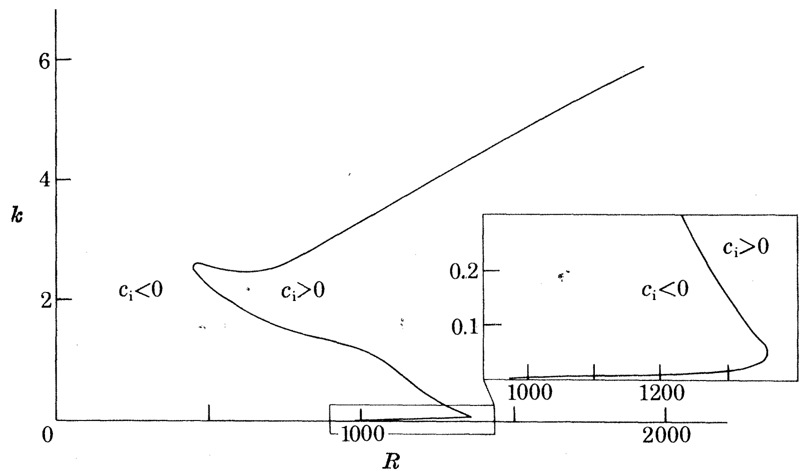

FIGURE 3. Neutral stability curve for surface waves in p.P.f. profile with $k_s = 7.3$.

those for p.P.f., while for $k_s = 14.6$ (figure 5) an island of instability appears between the wavenumber axis and the main part of the neutral curve. The origins of this island of instability can be seen in the calculations for small values of k_s, as the damping $(-c_i)$ has a local minimum for $R < R_c$. As k_s is increased this region of weak damping gives rise to the island of instability; further increases of k_s merely increase the size of the island, with no new islands appearing. The p.P.f. profiles were examined for any similar islands of instability, but none were found.

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The behaviour of the neutral curve for small k, as shown in figure 3, is typical of the results obtained for all the profiles examined. Initially R_n increases as k increases from zero, and in this range of k we expect the results of the limiting process (2.20) to apply. However, examination of §2.1.1 shows that the expansion (2.20) for c becomes invalid when $F \approx O(k^{-2})$, and hence the viscous long waves predicted in §2.1.1 exist only for $k < O\ (F^{-\frac{1}{2}})$. For air over water $F \approx 10^3\ k_s^3$. Thus we require $k < 0.03\ k_s^{-\frac{3}{2}}$ for the results of §2.1.1 to apply. For slightly larger k, gravitational effects outweigh viscous effects, and the long-wave solutions obtained numerically correspond to essentially inviscid long waves. In this range of k the value of R_n decreases as k increases and the neutral curve appears to be cusped near $k \approx 0\ (F^{-\frac{1}{2}})$. However, a close examination of the numerical results in this range showed that the transition from viscous to inviscid long waves was quite smooth.

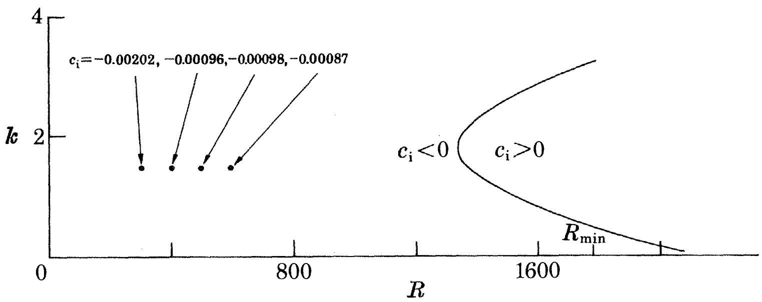

FIGURE 4. Neutral curve for surface waves in b.l.1 profile with $k_s = 7.3$. R_{min}, minimum value of R for long-wave instability.

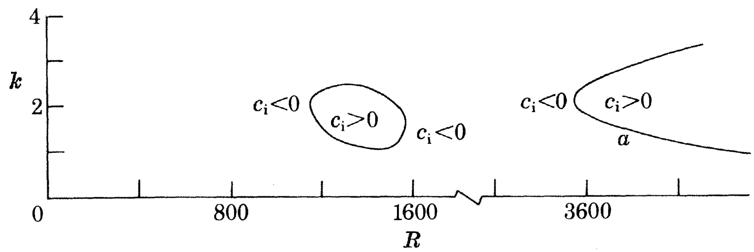

FIGURE 5. Neutral curve for surface waves in b.l.1 profile with $k_s = 14.6$. (a) Neutral curve obtained by starting from long-wave instability.

The results of §2.1.2 indicated that the eigenvalues of (2.29) can be separated into two groups, and clearly this separation will continue to apply for non-zero R. Thus the eigensolutions of (2.19) will represent either surface wave disturbances, associated with the two complex eigenvalues at $R = 0$, or shear wave disturbances that have developed from the purely imaginary eigenvalues at $R = 0$. The mode of instability defined by the neutral curves in figures 2–5 was found to be the surface-wave mode. Initially this was ascertained by tracing the development of both surface modes and shear modes as R increased from zero; invariably it was the downstream travelling surface-wave mode which first became unstable.

The surface wave is just a weakly damped (or growing) 'inviscid' water wave, the shearing air flow over the wave being responsible for transferring energy to the wave to overcome viscous dissipation (Miles 1957). The eigenfunction shown in figure 6 supports this view, as the stream function in the water is approximately that of an inviscid wave. However the characterization of the surface mode as an inviscid water wave is not completely correct as the phase speed of

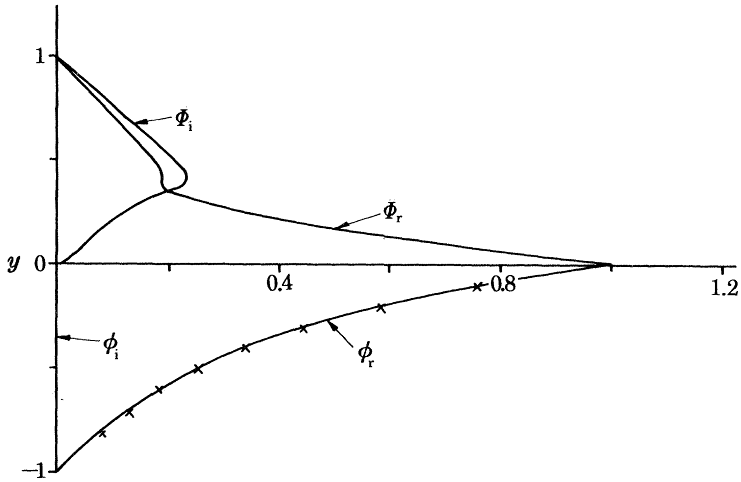

FIGURE 6. Eigenfunction at critical conditions for b.l.1 profile with $k_s = 58.4$. ×, Points on streamfunction for inviscid surface wave for $k = 2.643$.

TABLE 2. CRITICAL CONDITIONS FOR P.P.F. PROFILE

k_s	D_2/cm	R_c	k_c	c_r	inviscid wavespeed for $k = k_c$	$\mathscr{F}$	$\mathscr{T}$
3.647	1	380.56	1.8177	0.7112	0.457	0.322	0.0242
7.295	2	445.32	2.5426	1.173	0.904	1.88	0.0353
14.59	4	789.74	3.7504	1.446	1.16	4.78	0.0224
29.18	8	1835.3	4.042	1.619	1.33	7.08	0.00831
58.36	16	4263	4.627	1.798	1.51	10.5	0.00308

TABLE 3*a*. CRITICAL CONDITIONS FOR B.L.1 PROFILE

k_s	D_2/cm	R_c	k_c	c_r	inviscid wavespeed for $k = k_c$	$\mathscr{F}$	$\mathscr{T}$
3.647	1	589.87	1.4161	0.4059	0.311	0.134	0.0100
7.295	2	1341.4	1.7501	0.4434	0.343	0.207	0.00389
14.59	4	1142.3	1.9589	1.170	1.07	2.29	0.0107
29.18	8	2330.2	2.250	1.490	1.38	4.40	0.00516
58.36	16	5173	2.643	1.743	1.63	7.13	0.00209
116.7	32	12276	2.944	1.959	1.85	10.1	0.000743

TABLE 3*b*. CRITICAL CONDITIONS FOR B.L.1 PROFILE IN DIMENSIONAL VARIABLES

D_2 / cm	U_2 / cm s^{-1}	phase speed / cm s^{-1}	wavelength / cm	u_* / cm s^{-1}	shear stress / dyn cm^{-2}
1	242	34.7	4.44	8.20	0.0822
2	276	43.1	7.18	6.18	0.0477
4	117	48.4	12.85	2.85	0.00995
8	120	62.9	22.4	2.04	0.00507
16	133	81.8	38.1	1.52	0.00282
32	158	108.9	68.4	1.17	0.00167

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the surface mode is always greater than the phase speed of the corresponding inviscid wave (see tables 2–5).

TABLE 4*a*. CRITICAL CONDITIONS FOR B.L.2 PROFILE

k_s	D_2/cm	R_c	k_c	c_r	inviscid wavespeed for $k = k_c$	$\mathscr{F}$	$\mathscr{T}$
3.647	1	269.83	0.87464	1.180	0.852	0.640	0.0481
7.295	2	293.85	1.3652	2.152	1.80	4.32	0.0810
14.59	4	674.69	1.5437	2.424	2.06	6.55	0.0307
29.18	8	1581.8	1.843	2.641	2.28	9.54	0.0112
58.36	16	3851	2.201	2.790	2.42	12.9	0.00377

TABLE 4*b*. CRITICAL CONDITIONS FOR B.L.2 PROFILE IN DIMENSIONAL VARIABLES

D_2/cm	U_2/cm s^{-1}	phase speed/cm s^{-1}	wavelength/cm	u_*/cm s^{-1}	shear stress/dyn cm^{-2}†
1	141	46.1	7.18	6.05	0.00445
2	74.2	44.3	9.22	3.09	0.00117
4	87.9	59.2	16.3	2.38	0.00696
8	103	75.8	26.2	1.83	0.00411
16	126	97.4	45.6	1.42	0.00248

† dyn = 10^{-5} N

TABLE 5. CRITICAL CONDITIONS FOR P.C.F. PROFILE

k_s	D_2/cm	R_c	k_c	c_r	inviscid wavespeed for $k = k_c$	$\mathscr{F}$	$\mathscr{T}$
3.647	1	849.42	1.5002	0.2556	0.213	0.0646	0.00485
14.59	4	5650.0	2.2810	0.2502	0.202	0.0934	0.000438

Neutral conditions for the shear wave mode could only be found for the p.P.f. profile. Figure 2 shows the neutral curves obtained for both the surface and shear modes. For the shear wave in the air the interface behaves effectively as a rigid boundary moving with velocity $U_b(0)$, and hence we can use the results of Reynolds & Potter (1967) for combined Poiseuille–Couette flow to estimate the critical conditions. Taking into account the different scaling, we find that, in the notation of Reynolds & Potter, the quantity $u_W = 0.092$ for the air. Thus, in our scaling, disturbances confined to the air would first become unstable when $R \approx 12{,}000$ and $k \approx 1.6$. As the water surface has been approximated by a rigid boundary, the stability of the air flow is then independent of gravitational and surface tension effects, and hence this estimate of the critical conditions for shear waves applies to the p.P.f. calculation for all values of k_s. The results in figure 2 show that there is fair agreement between the estimated and calculated critical wavenumbers, while the estimated critical Reynolds number is almost 30% higher than the calculated critical value, $R \approx 8800$. This is fair agreement given the drastic nature of the approximations involved.

For both b.l.1 and b.l.2 the Reynolds & Potter parameter $u_W = 3$, and hence the shear wave disturbances in the air will be damped for all R. Similarly, in the p.C.f. profile, where the velocity distribution in the air is linear, the shear wave disturbances are always damped. Also, in the cases examined, the surface wave, which for small R travelled upstream, always remained damped. Thus, for the range of parameters considered, the stability of the p.P.f., b.l. and p.C.f. profiles was determined by the downstream travelling surface-wave mode.

Tables 2–5 give the critical conditions, as a function of k_s, for the profiles studied. Included in the tables are the values of $\mathscr{F}$ and $\mathscr{T}$ at critical conditions, where these parameters have been calculated by using (2.15) and (2.16). The non-dimensional constants at critical conditions give little physical feeling for the problem, and so critical conditions for the b.l. profiles are given in dimensional quantities in tables 3*b* and 4*b*.

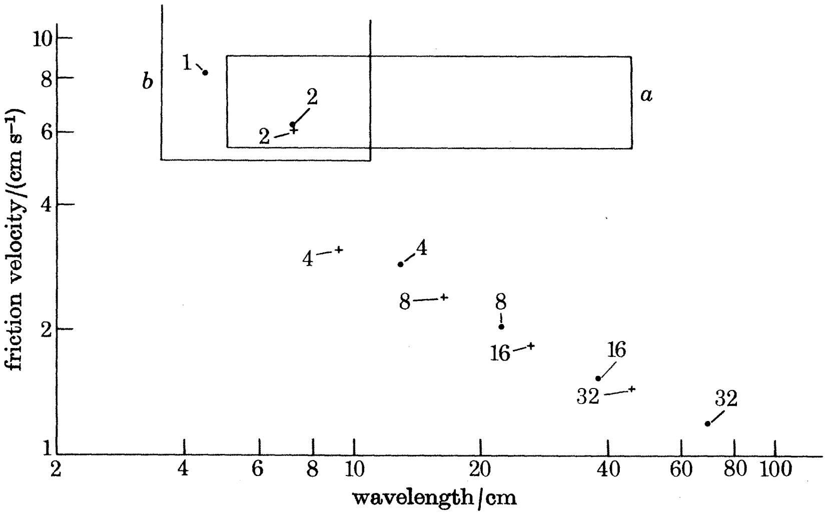

FIGURE 7. Critical conditions for b.l.1 and b.l.2 compared with experimental conditions used in other work. (*a*) Wilson *et al.* (1973); (*b*) Gottfredi & Jameson (1970). •, b.l.1 profile; +, b.l.2 profile. Depth of water/cm shown at each point.

Comparison of these results with experimental results is difficult as, to date, there have been no experimental investigations of the stability of the laminar flow of air over water; invariably experimental investigations have examined wave growth (or decay) due to a turbulent air flow. However it is still possible to make some qualitative comparisons with published experimental results. Figure 7 shows the friction velocity u_*, as a function of wavelength, at critical conditions. Superimposed on this graph are the regions of u_*/wavelength space considered by Gottifredi & Jameson (1970) and Wilson *et al.* (1973). (The friction velocity is defined by

$$u_*^2 = \frac{\text{shear stress at interface}}{\text{density of air}}$$

and figure 7 is adapted from figure 2 of Larsen & Wright 1975.)

Gottifredi & Jameson (1970) claim that the minimum value of u_* for wave growth is about 5 cm s^{-1} for wavelengths of 4–8 cm, while from Wilson *et al.* (1973) it can be inferred that the minimum value of u_* is about 8 cm s^{-1} for the same range of wavelengths. Thus our laminar flow calculations lead to values of u_* similar to those determined by experiment. However, the agreement with experiment is not as good as it first seems. The conditions that correspond to the experimental apparatus of Gottifredi & Jameson occur for b.l.2 with $k_s = 14.6$, i.e. a water depth of 8 cm (Gottifredi & Jameson had a water depth of *ca.* 7 cm), for which $u_* \approx 2.5$ cm s^{-1}, with a wavelength of 16 cm at critical conditions. Similarly, for Wilson *et al.* (1973), who used a water depth of *ca.* 46 cm, we would estimate a value of $u_* \lesssim 1.5$ cm s^{-1} at critical conditions.

It is interesting to note that the critical conditions that fall below the experimentally investigated régimes are all associated with the presence of the island of instability. As experimental work has tended to examine the growth rate of waves for a fixed wind speed, it is possible that growing long waves at small values of u_* have not been investigated. Indeed, Wilson *et al.* (1973) examined wind speeds of 20 cm s^{-1}, 112 cm s^{-1} and 180 cm s^{-1}, while for b.l.2 we find that critical conditions can exist with a free-stream speed as small as 75 cm s^{-1}.

TABLE 6. WAVENUMBERS FOR EQUAL DISSIPATION IN BOTTOM BOUNDARY LAYER AND BULK OF FLUID

depth of water cm	k/d
1	1.70
2	1.90
4	2.10
8	2.35
16	2.55
32	2.80

From the results for the profile b.l.1 it is evident that the finite depth of the water has a significant effect on the determination of the critical conditions. In fact all the critical wavenumbers are such that the energy dissipation due to the viscous layer on the bottom is of the same order as that due to the approximately irrotational straining in the bulk of the fluid. The ratio of dissipation at the bottom to the dissipation in the bulk of the fluid can be expressed as (from Phillips 1969, ch. 3)

$$\left(g\,\frac{k}{d}\right)^{\frac{1}{4}} D_1^{\frac{3}{4}} \Big/ 4(2\nu)^{\frac{1}{2}} \left(\frac{k}{d}\right) \sinh^2\left(\frac{k}{d}\right),$$

and the values of k/d at which this quantity is equal to unity are given in table 6.

For b.l.1 these wavenumbers are close to the critical wavenumbers in table 3, while in b.l.2 (where $d = 0.5$) the critical wavenumbers are all greater than the wavenumbers given in table 6. Thus in b.l.2 the shearing air flow has only to supply sufficient energy to overcome the damping due to the bulk straining, thus leading to lower u_* and lower free-stream velocities for the initiation of the waves. Profiles with smaller values for d were not considered as the interface velocity increases as d decreases. In b.l.1 $u_b(0) \approx 5\,\%$ of the free-stream speed and in b.l.2 $u_b(0) \approx 10\,\%$ of the free-stream speed. Any further decrease in the depth ratio would make the basic flow too crude a model of both experimental and oceanic situations.

Another feature of the results for all the profiles studied is the prediction of rather large wavelengths at critical conditions. Surface tension effects are important for $k > k_s$, but all the critical wavenumbers are less than k_s. Also, as the depth of water increases k_c becomes much smaller than k_s and hence surface tension has little effect on the critical conditions. This is in contrast to most observations where small scale capillary-gravity waves tend to appear. This difference may be due to the fact that under experimental conditions with turbulent air flow the Phillips (1957) mechanism may produce the small-scale waves, whereas in our laminar flow profiles this mechanism is absent.

2.3. The nonlinear analysis

The results of the linear stability calculations for air over water have shown that the stability of the basic flow depends on the growth or decay of surface-wave modes. For these modes a critical point (R_c, k_c) with $k_c \neq 0$ exists, and the neutral curve in this region is similar to the neutral curve for the plane Poiseuille flow of a homogeneous fluid. Thus we can apply the theory of Stewartson & Stuart (1971) (hereafter referred to as S.S.) to calculate the nonlinear effects on the growth of surface waves. Before proceeding with the nonlinear calculation it is first necessary to obtain more information about the structure of the disturbance at critical conditions, and we need to define a problem that is the adjoint of the system (2.19).

2.3.1. *The adjoint system*

If M is an ordinary differential operator over a region N, then the adjoint problem can be defined by

$$\int_N \psi M(\phi)\, \mathrm{d}y = \int_N \phi M^+(\psi)\, \mathrm{d}y = 0.$$

However, we are interested in a problem where N consists of two sub-regions, and the way in which the contributions from the sub-regions should be combined is not clear. This difficulty can be avoided by using a formulation of the problem that involves variables that are continuous at the interface of the sub-regions so that, effectively, we have only one region to consider.

Thus we define the quantities

$$\left.\begin{aligned}
V(y) &= \begin{cases} -\mathrm{i}k\Phi & \text{for } 0 < y < 1,\\ -\mathrm{i}k\phi & \text{for } -d^{-1} < y < 0,\end{cases}\\
U(y) &= \begin{cases} \Phi' - \Phi U_b'/\sigma(y) & \text{for } 0 < y < 1,\\ \phi' - \phi u_b'/\sigma(y) & \text{for } -d^{-1} < y < 1,\end{cases}\\
\tau(y) &= \begin{cases} (\Phi'' + k^2\Phi)/(\mu(y)\, R) & \text{for } 0 < y < 1,\\ (\phi'' + k^2\phi)/(\mu(y)\, R) & \text{for } -d^{-1} < y < 1,\end{cases}\\
n(y) &= \Phi\left[\frac{F + k^2 T(y)}{\rho(y)\,\sigma(y)} + \frac{U_b'}{\rho(y)}\right] - \Phi'\left[\frac{3k^2}{\mathrm{i}k\mu(y)\,R} + \frac{\sigma(y)}{\rho(y)}\right] + \frac{\Phi'''}{\mathrm{i}k\mu(y)\,R} \quad \text{for } 0 < y < 1\\
\text{and}\quad n(y) &= \phi\left[\frac{F + k^2 T(y)}{\rho(y)\,\sigma(y)} + \frac{u_b'}{\rho(y)}\right] - \phi'\left[\frac{3k^2}{\mathrm{i}k\mu(y)\,R} + \frac{\sigma(y)}{\rho(y)}\right] + \frac{\phi'''}{\mathrm{i}k\mu(y)\,R} \quad \text{for } -d^{-1} < y < 1,
\end{aligned}\right\} \tag{2.30}$$

where
$$\sigma(y) = \begin{cases} U_b(y) - c & \text{for } 0 < y < 1,\\ u_b(y) - c & \text{for } -d^{-1} < y < 0,\end{cases} \tag{2.31a}$$

and the step functions $T(y)$, $\rho(y)$ and $\mu(y)$ are given by

$$T(y) = \begin{cases} 0 \\ SD_2/\rho_1 v_2^2,\end{cases} \quad \rho(y) = \begin{cases} 1 \\ \rho_2/\rho_1\end{cases} \quad \text{and} \quad \mu(y) = \begin{cases} 1 & \text{for } 0 < y < 1,\\ \mu_2/\mu_1 & \text{for } -d^{-1} < y < 0.\end{cases} \tag{2.31b, c, d}$$

Here V, U, τ and n are the vertical velocity, horizontal velocity, shear stress and normal stress on a material surface respectively, and hence are continuous at the interface. If $W^T = [V, U, \tau, n]$ (W^T is the transpose of W), then (2.19) can be written as

$$W' = SW, \tag{2.32a}$$

$$S = \begin{bmatrix} \sigma'/\sigma & -\mathrm{i}k & 0 & 0 \\ (k^2+\sigma''/\sigma)/\mathrm{i}k & \sigma'/\sigma & \mu R & 0 \\ \dfrac{4\mathrm{i}k\sigma'}{\mu\sigma R}+\dfrac{F+k^2T}{\rho\sigma R^2} & \dfrac{4k^2}{\mu R}+\dfrac{\mathrm{i}k\sigma}{\rho} & 0 & \mathrm{i}k \\ -\mathrm{i}k\sigma/\rho & \dfrac{F+k^2T}{\rho\sigma R^2} & \mathrm{i}k & 0 \end{bmatrix}. \tag{2.32b}$$

The boundary conditions for (2.32) are

$$V = U = 0 \quad \text{on} \quad y = -d^{-1} \quad \text{and} \quad y = 1 \tag{2.33}$$

and W is continuous at the interface in the sense that

$$W(0^-) = W(0^+).$$

We note that the elements of S are only piecewise continuous functions, with finite jumps at $y = 0$, and hence W' also has finite discontinuities at the interface. Thus the problem is in the form to which the usual theory of adjoints applies (Coddington & Levinson 1955). If $(W^+)^{\mathrm{T}} = [n^+, \tau^+, U^+, V^+]$ is the transpose of the adjoint function, then we require

$$\int_{-d^{-1}}^{1} (W^+)^{\mathrm{T}} (W'-SW)\,\mathrm{d}y = (W^+)^{\mathrm{T}}\, W\Big|_{-d^{-1}}^{1} - \int_{-d^{-1}}^{1} W^{\mathrm{T}}\,[(W^+)'+S^{\mathrm{T}}W^+]\,\mathrm{d}y = 0.$$

The contribution from the boundaries is zero if $U^+ = V^+ = 0$ on $y = -d^{-1}$ and $y = 1$, and so writing $S^+ = -S^{\mathrm{T}}$, the adjoint system becomes

$$\left.\begin{aligned} (W^+)' &= S^+W^+, \\ U^+ = V^+ = 0 \quad &\text{on} \quad y = -d^{-1}, \quad y = 1, \\ W^+ \quad &\text{continuous at} \quad y = 0. \end{aligned}\right\} \tag{2.34}$$

Although the matrix form of the problem and its adjoint is very compact, it is more convenient to work with a single fourth-order equation for the nonlinear analysis and for the computational work. Thus if

$$V^+ = \begin{cases} \Psi & \text{for} \quad 0 < y < 1, \\ \psi & \text{for} \quad -d^{-1} < y < 0, \end{cases}$$

then the adjoint system, at critical conditions is

$$\left.\begin{aligned} &\Psi = \Psi' = 0 \quad \text{on} \quad y = 1, \\ &L^+(\Psi) = \mathrm{i}k_c(U_b-c_r)\,(\Psi''-k_c^2\Psi)+2\mathrm{i}k_c U_b'\Psi' - R_c^{-1}(\Psi^{(\mathrm{iv})}-2k_c^2\Psi''+k_c^4\Psi) = 0, \\ &\left.\begin{aligned} &\psi = \Psi, \quad \psi' = \Psi', \\ &\psi''+k_c^2\psi = \mu(\Psi''+k_c^2\Psi), \\ &-\frac{\psi'''-3k_c^2\psi'}{\nu R_c}+\frac{u_b'(\psi''+k_c^2\psi)}{\sigma_r \nu R_c}+\mathrm{i}k_c\sigma_r\psi'-\frac{\mathrm{i}k_c\psi(F+k_c^2T)}{\sigma_r R_c^2} \\ &\quad = \rho\left[-\frac{\Psi'''-3k_c^2\Psi'}{R_c}+\frac{U_b'(\Psi''+k_c^2\Psi)}{\sigma_r R_c}+\mathrm{i}k_c\sigma_r\Psi'-\frac{\mathrm{i}k_c\Psi F}{\sigma_r R_c^2}\right], \end{aligned}\right] \text{on} \quad y = 0 \\ &l^+(\psi) = \mathrm{i}k_c(u_b-c_r)\,(\psi''-k_c^2\psi)+2\mathrm{i}k_c u_b'\psi' - (\nu R_c)^{-1}\,(\psi^{(\mathrm{iv})}-2k_c^2\psi''+k_c^4\psi) = 0, \\ &\psi = \psi' = 0 \quad \text{on} \quad y = -d^{-1}, \end{aligned}\right\} \tag{2.35}$$

where $\sigma_r = u_b(0) - c_r = U_b(0) - c_r$ and ρ, μ, ν and T are the constants defined in §2.1. As expected, the adjoint equation in each fluid layer is the same as the usual adjoint of the Orr–Sommerfeld equation for a homogeneous fluid. In this formulation of the adjoint system we have the relation

$$\int_{-d^{-1}}^{0} \psi l(\phi)\, dy + \rho \int_{0}^{1} \Psi L(\Phi)\, dy = \int_{-d^{-1}}^{0} \phi l^{+}(\psi)\, dy + \rho \int_{0}^{1} \Phi L^{+}(\Psi)\, dy = 0, \tag{2.36}$$

where L and l are the Orr–Sommerfeld operators for the upper and lower fluids in (2.19).

2.3.2. *The critical constants*

Further properties of the disturbance near critical conditions can be found, by following S.S., by expanding the frequency and eigenfunction as a Taylor series about the critical conditions. Thus we write

$$-ikc = -ik_c c_r + ia_1(k-k_c) - a_2(k-k_c)^2 + d_1(R-R_c) + \dots, \tag{2.37}$$

where a_1 is a real number, and a_2 has positive real part. Since $d(kc)/dk|_{k_c} = -a_1$, we see that $-a_1$ is the group velocity of the disturbance; a_2 is related to the curvature of the nose of the neutral curve and d_1 represents the exponential growth of the wave for $R > R_c$. For the stream function formulation (2.19) we use the expansion

$$\Phi = \Phi_1 + (k-k_c)\, \Phi_{10} + (R-R_c)\, \Phi_{11} + (k-k_c)^2\, \Phi_{12} + \dots \tag{2.38}$$

with a similar expansion for ϕ in the lower fluid, while for the matrix formulation (2.32) we write

$$\left.\begin{aligned} S &= S_1 + (k-k_c)\, S_{10} + (R-R_c)\, S_{11} + (k-k_c)^2\, S_{12} + \dots \\ \text{and} \quad W &= W_1 + (k-k_c)\, W_{10} + (R-R_c)\, W_{11} + (k-k_c)\, W_{12} + \dots. \end{aligned}\right\} \tag{2.39}$$

Substituting (2.39) in (2.32*a*) and equating like powers of $k-k_c$ and $R-R_c$, we obtain at $O(1)$,

$$\left.\begin{aligned} W_1' - S_1 W_1 &= 0, \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.40}$$

where S_1 is the matrix S given by (2.32*b*) with R, k and c taking the values R_c, k_c and c_r respectively. The adjoint to (2.40) is

$$\left.\begin{aligned} (W_c^{+})' - (-S_c^{T})\, W_c^{+} &= 0, \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.41}$$

where W_c^{+} is the adjoint function at critical conditions.

At $O(k-k_c)$ we get the system

$$\left.\begin{aligned} W_{10}' - S_1 W_{10} &= S_{10} W_1, \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.42}$$

where S_{10} is given in appendix B. For (2.42) to have a solution the forcing terms must be orthogonal to the adjoint functio n, i.e.

$$\int_{-d^{-1}}^{1} (W_c^{+})\, S_{10} W_1\, dy = 0. \tag{2.43}$$

The matrix S_{10} has the form $(A+a_1B)$, and so (2.43) is an equation defining a_1. In terms of the stream function (2.43) is equivalent to the result

$$\begin{aligned} a_1(CDEN) = {} & \frac{\mathrm{i}c_\mathrm{r}\phi_1}{\nu k_\mathrm{c}R_\mathrm{c}\sigma_\mathrm{r}^2}(u_\mathrm{b}'-U_\mathrm{b}')(\psi''+k_\mathrm{c}^2\psi)-\frac{2\mathrm{i}k_\mathrm{c}\phi_1\psi'(1-\mu)}{\nu R_\mathrm{c}} \\ & -\psi\left[\phi_1u_\mathrm{b}'-\phi_1'u_\mathrm{b}-\rho(\Phi_1U_\mathrm{b}'-\Phi_1'U_\mathrm{b})+\frac{6\mathrm{i}k_\mathrm{c}}{\nu R_\mathrm{c}}(\phi_1'-\mu\Phi_1')\right. \\ & \left.+\phi_1\frac{F(u_\mathrm{b}-2c_\mathrm{r})(1-\rho)+k_\mathrm{c}^2T(3u_\mathrm{b}-4c_\mathrm{r})}{\sigma_\mathrm{r}^2R_\mathrm{c}^2}\right] \\ & +\int_{-d^{-1}}^{0}\psi[(4\mathrm{i}k_\mathrm{c}/\nu R_\mathrm{c}-u_\mathrm{b})(\phi_1''-k_\mathrm{c}^2\phi_1)+\phi_1(2k_\mathrm{c}^2\sigma+u_\mathrm{b}'')]\,\mathrm{d}y \\ & +\rho\int_0^1\Psi[(4\mathrm{i}k_\mathrm{c}/R_\mathrm{c}-U_\mathrm{b})(\Phi_1''-k_\mathrm{c}^2\Phi_1)+\Phi_1(2k_\mathrm{c}^2\sigma+U_\mathrm{b}'')]\,\mathrm{d}y, \end{aligned} \tag{2.44}$$

where the mnemonic $CDEN$ is given by

$$\begin{aligned} CDEN = {} & \frac{-\mathrm{i}\phi_1}{\nu k_\mathrm{c}R_\mathrm{c}\sigma_\mathrm{r}^2}(u_\mathrm{b}'-U_\mathrm{b}')(\psi''+k_\mathrm{c}^2\psi)+\psi\left[\rho\Phi_1'-\phi_1'-\phi_1\frac{F(1-\rho)+k_\mathrm{c}^2T}{\sigma_\mathrm{r}^2R_\mathrm{c}^2}\right] \\ & +\int_{-d^{-1}}^{0}\psi(\phi_1''-k_\mathrm{c}^2\phi_1)\,\mathrm{d}y+\rho\int_0^1\Psi(\Phi_1''-k_\mathrm{c}^2\Phi_1)\,\mathrm{d}y. \end{aligned} \tag{2.45}$$

In (2.44) and (2.45) all quantities that are not integrated are evaluated at $y = 0$; this convention will also be used in the definitions of a_2, d_1 and κ. This form for a_1 is analogous to the definition given by (2.21) of S.S. for homogeneous fluids; here we have extra contributions due to the presence of the interface.

Similarly at $O(R-R_\mathrm{c})$ we obtain the system

$$\left.\begin{aligned} W_{11}'-S_1W_{11} &= S_{11}W_1, \\ \text{no slip on boundaries,} & \end{aligned}\right\} \tag{2.46}$$

where S_{11} is given in appendix B. The solvability condition for (2.46) is

$$\int_{-d^{-1}}^{1}(W_\mathrm{c}^+)^\mathrm{T}S_{11}W_1\,\mathrm{d}y = 0, \tag{2.47}$$

which defines d_1. In terms of the stream function we have

$$\begin{aligned} d_1(CDEN) = {} & \frac{\mathrm{i}k_\mathrm{c}\psi}{R_\mathrm{c}}\left[\phi_1\frac{F(1-\rho)+k_\mathrm{c}^2T}{\sigma_\mathrm{r}R_\mathrm{c}^2}+(1-\rho)(\phi_1'\sigma_\mathrm{r}-\phi_1u_\mathrm{b}')\right] \\ & -\frac{1}{\nu R_\mathrm{c}^2}\int_{-d^{-1}}^{0}\psi(\phi_1^{(\mathrm{iv})}-2k_\mathrm{c}^2\phi_1''+k_\mathrm{c}^4\phi_1)\,\mathrm{d}y-\frac{\rho}{R_\mathrm{c}^2}\int_0^1\Psi(\Phi_1^{(\mathrm{iv})}-2k_\mathrm{c}^2\Phi_1''+k_\mathrm{c}^4\Phi_1)\,\mathrm{d}y. \end{aligned} \tag{2.48}$$

Finally, at $O(k-k_\mathrm{c})^2$, we have

$$\left.\begin{aligned} W_{12}'-S_1W_{12} &= S_{10}W_{10}+S_{12}W_1, \\ \text{no slip on boundaries,} & \end{aligned}\right\} \tag{2.49}$$

where S_{12} is given in appendix B. The solvability condition

$$\int_{-d^{-1}}^{1}(W_\mathrm{c}^+)^\mathrm{T}[S_{10}W_{10}+S_{12}W_1]\,\mathrm{d}y = 0 \tag{2.50}$$

now defines a_2, which in terms of the streamfunction is given by

$$
\begin{aligned}
a_2(CDEN) = & -\frac{\psi''+k_c^2\psi}{\nu R_c}(u_b'-U_b')(a_1+c_r)\left[\frac{\phi_{10}}{k_c\sigma_r^2}-\frac{\phi_1(u_b+a_1)}{k_c^2\sigma_r^3}\right]+\frac{\psi'}{\nu R_c}(1-\mu)(2k_c\phi_{10}+\phi_1) \\
& -\psi\left\{\mathrm{i}[\phi_{10}u_b'-\phi_{10}'(u_b+a_1)]-\mathrm{i}\rho[\Phi_{10}U_b'-\Phi_{10}'(U_b+a_1)]-\frac{6k_c}{\nu R_c}(\phi_{10}'-\mu\Phi_{10}')\right. \\
& +\mathrm{i}\phi_{10}\frac{F(u_b-2c_r-a_1)(1-\rho)+k_c^2T(3u_b-4c_r-a_1)}{\sigma_r^2R_c^2} \\
& \left.+\mathrm{i}\phi_1\left[(a_1+c_r)^2\frac{F(1-\rho)+k_c^2T}{\sigma_r^3k_cR_c^2}+\frac{k_cT}{\sigma_r^2R_c^2}(3u_b-5c_r-2a_1)\right]-\frac{3}{\nu R_c}(\phi_1'-\mu\Phi_1')\right\} \\
& -\int_{-d^{-1}}^{0}\psi\left\{\phi_{10}\left[-\mathrm{i}u_b''-\frac{4k_c^3}{\nu R_c}-\mathrm{i}k_c^2(u_b+a_1)-2\mathrm{i}k_c^2\sigma\right]+\phi_{10}''\left[\frac{4k_c}{\nu R_c}+\mathrm{i}(u_b+a_1)\right]\right. \\
& \left.+\phi_1\left[-\frac{6k_c^2}{\nu R_c}-\mathrm{i}k_c\sigma-2\mathrm{i}k_c(u_b+a_1)\right]+\phi_1''\frac{2}{\nu R_c}\right\}\mathrm{d}y \\
& -\rho\int_0^1\Psi\left\{\Phi_{10}\left[-\mathrm{i}U_b''-\frac{4k_c^3}{R_c}-\mathrm{i}k_c(U_b+a_1)-2\mathrm{i}k_c^2\sigma\right]+\Phi_{10}''\left[\frac{4k_c}{R_c}+\mathrm{i}(U_b+a_1)\right]\right. \\
& \left.+\Phi_1\left[-\frac{6k_c^2}{R_c}-\mathrm{i}k_c\sigma-2\mathrm{i}k_c(U_b+a_1)\right]+\Phi_1''\frac{2}{R_c}\right\}\mathrm{d}y. \qquad (2.51)
\end{aligned}
$$

The systems of $O(1)$, $O(k-k_c)$, $O(R-R_c)$ and $O(k-k_c)^2$ for the streamfunction formulation are given in appendix B; the critical constants can also be derived from these systems by using the adjoint relation (2.36), and this forms a useful check on the foregoing calculations.

2.3.3. *Derivation of the amplitude equation*

The governing equations and boundary conditions for the two-dimensional motion of two superposed fluids are

$$
\left.
\begin{aligned}
& U = V = 0 \quad \text{on} \quad y=1, \\
& \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x}+R^{-1}\left(\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{\partial y^2}\right), \\
& \frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y}+R^{-1}\left(\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial y^2}\right), \\
& \frac{\partial U}{\partial x}+\frac{\partial V}{\partial y} = 0, \\
& \left.
\begin{aligned}
& u=U, \quad v=V, \quad \frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}=v, \quad \frac{\partial\eta}{\partial t}+U\frac{\partial\eta}{\partial x}=V, \\
& 2\eta_x\frac{\partial v}{\partial y}+\tfrac12(1-\eta_x^2)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=\mu\left[2\eta_x\frac{\partial V}{\partial y}+\tfrac12(1-\eta_x^2)\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right], \\
& \rho p-\frac{F}{R^2}\eta-\frac{2}{\nu R(1+\eta_x^2)}\left[(1-\eta_x^2)\frac{\partial v}{\partial y}-\eta_x\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{T\eta_{xx}}{R^2(1+\eta_x^2)^{\frac32}} \\
& \quad =\rho\left\{P-\frac{F}{R^2}\eta-\frac{2}{R(1+\eta_x^2)}\left[(1-\eta_x^2)\frac{\partial V}{\partial y}-\eta_x\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)\right]\right\},
\end{aligned}
\right\} \text{on} \quad y=\eta(x,t), \\
& \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}(\rho p)+(\nu R)^{-1}\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}\right), \\
& \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial}{\partial y}(\rho p)+(\nu R)^{-1}\left(\frac{\partial^2v}{\partial x^2}+\frac{\partial^2v}{\partial y^2}\right), \\
& \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \\
& u = v = 0 \quad \text{on} \quad y=-d^{-1}.
\end{aligned}
\right\} \qquad (2.52)
$$

The linear stability analysis of the preceding sections has shown that the basic flow becomes unstable to travelling wave disturbances when $R = R_c$. We now wish to examine the growth of such waves, taking account of nonlinear and dispersive effects, in a region near R_c. Thus following S.S., we introduce the small parameter ϵ such that

$$\epsilon = d_{1r}|R-R_c| \quad \text{as} \quad R \to R_c, \tag{2.53}$$

and the scaled variables τ and ξ given by

$$\tau = \epsilon t, \tag{2.54a}$$

and
$$\xi = \epsilon^{\frac{1}{2}}(x+a_1 t). \tag{2.54b}$$

The equations of continuity in (2.52) are satisfied by introducing the streamfunctions Λ and λ for the upper and lower fluids. The velocities are then given by

$$U = \frac{\partial \Lambda}{\partial y} \quad \text{and} \quad V = -\frac{\partial \Lambda}{\partial x}, \tag{2.55}$$

with similar definitions for the lower fluid. Following S.S. we introduce the expansions

$$\lambda = \phi_0(y, \xi, \tau) + [\phi_1(y, \xi, \tau)\, E + \phi_2(y, \xi, \tau)\, E^2 + \text{c.c.}] + \ldots, \tag{2.56a}$$

where c.c. denotes complex conjugate, and where

$$E = \exp\,[\mathrm{i}k_c(x-c_r t)]. \tag{2.56b}$$

The functions in (2.56a) are further expanded as follows:

$$\phi_0(y, \xi, \tau) = \int_0^y u_b\,\mathrm{d}y + \epsilon\phi_{02}(y, \xi, \tau) + O(\epsilon^{\frac{3}{2}}), \tag{2.56c}$$

$$\phi_1(y, \xi, \tau) = \epsilon^{\frac{1}{2}}\phi_{11}(y, \xi, \tau) + \epsilon\phi_{12}(y, \xi, \tau) + \epsilon^{\frac{3}{2}}\phi_{13}(y, \xi, \tau) + \ldots \tag{2.56d}$$

and
$$\phi_2(y, \xi, \tau) = \epsilon\phi_{22}(y, \xi, \tau) + O(\epsilon^{\frac{3}{2}}). \tag{2.56e}$$

The pressure and interface displacement are expanded as

$$p = p_0(x, y, \xi, \tau) + [p_1(y, \xi, \tau)\, E + p_2(y, \xi, \tau)\, E^2 + \text{c.c.}] + \ldots \tag{2.57}$$

and
$$\eta = \theta_0(\xi, \tau) + [\theta_1(\xi, \tau)\, E + \theta_2(\xi, \tau)\, E^2 + \text{c.c.}] + \ldots, \tag{2.58}$$

where
$$p_0(x, y, \xi, \tau) = \frac{u_b'' x}{\mu R_c} + \epsilon^{\frac{1}{2}} p_{01}(\xi, \tau) + \epsilon\left[p_{02}(y, \xi, \tau) - \frac{u_b'' x}{\mu R_c^2 d_{1r}}\right] + O(\epsilon^{\frac{3}{2}}) \tag{2.59}$$

and
$$\theta_0(\xi, \tau) = \epsilon\theta_{02}(\xi, \tau) + O(\epsilon^{\frac{3}{2}}). \tag{2.60}$$

The terms remaining in (2.57) and (2.58) are expanded as the corresponding terms in (2.56), and for the streamfunction and pressure in the upper fluid we have expansions similar to (2.56) and (2.57) respectively.

The expansions (2.56) are substituted into the equations of motion and like powers of ϵE collected to give the equations governing the disturbance, the distortion of the mean flow, the production of the second harmonic, and finally the distortion of the disturbance. However, as the interface conditions are applied on $y = \eta(x, t)$, we must first express U, V and P and their derivatives as Taylor series about $y = 0$, keeping terms that are quadratic in η, before using the expansions (2.56–60). The interface conditions for the different equations are then obtained by collecting like powers of ϵE. The whole process of obtaining the interface conditions involves some extremely lengthy algebra, and so only the results will be given. Thus at $O(\epsilon^{\frac{1}{2}}E)$ we obtain the coupled Orr–Sommerfeld problem (where, as before, a prime denotes differentiation with respect to y):

$$\left.\begin{aligned}
&\Phi_{11} = \Phi_{11}' = 0 \quad \text{on} \quad y = 1,\\
&L(k_c, \Phi_{11}) = ik_c(U_b - c_r)(\Phi_{11}'' - k_c^2\Phi_{11}) - ik_c U_b''\Phi_{11} - R_c^{-1}(\Phi_{11}^{(iv)} - 2k_c^2\Phi_{11}'' + k_c^4\Phi_{11}) = 0,\\
&\left.\begin{aligned}
&\phi_{11} = \Phi_{11},\\
&\phi_{11}' - u_b'\phi_{11}/\sigma_r = \Phi_{11}' - U_b'\Phi_{11}/\sigma_r, \quad \phi_{11}'' + k_c^2\phi_{11} = \mu(\Phi_{11}'' + k_c^2\Phi_{11}),\\
&q(k_c, \phi_{11}) = (\nu R_c)^{-1}(\phi_{11}''' - 3k_c^2\phi_{11}') + ik_c(\phi_{11}u_b' - \phi_{11}'\sigma_r)\\
&\qquad\qquad + ik\phi_{11}(F + k_c^2 T)/\sigma_r R_c^2\\
&\quad = \rho[R_c^{-1}(\Phi_{11}''' - 3k_c^2\Phi_{11}') + ik_c(\Phi_{11}U_b' - \Phi_{11}'\sigma_r) + ik_c\Phi_{11}F/\sigma_r R_c^2]\\
&\quad = \rho Q(k_c, \Phi_{11}),
\end{aligned}\right] \text{on} \quad y = 0\\
&l(k_c, \phi_{11}) = ik_c(u_b - c_r)(\phi_{11}'' - k_c^2\phi_{11}) - ik_c u_b''\phi_{11} - (\nu R_c)^{-1}(\phi_{11}^{(iv)} - 2k_c^2\phi_{11}'' + k_c^4\phi_{11}) = 0,\\
&\phi_{11} = \phi_{11}' = 0 \quad \text{on} \quad y = -d^{-1},
\end{aligned}\right\} \tag{2.61}$$

which is just (2.19) with $R = R_c$, $k = k_c$ and $c = c_r$. The solution of (2.61) is written as

$$\left.\begin{aligned}
\Phi_{11} &= A(\tau, \xi)\,\Psi_1(y),\\
\phi_{11} &= A(\tau, \xi)\,\psi_1(y),
\end{aligned}\right\} \tag{2.62}$$

where the eigenfunctions are normalized to unity at $y = 0$ and $A(\tau, \xi)$ is the amplitude function to be determined.

At this stage it is convenient to introduce a notation to simplify the ensuing equations; the shorthand form is based on the close similarity of expressions, for the same quantity, in both the lower and upper fluids. Thus the expression

$$Q(k_c, \Phi_{11}) = [q(k_c, \phi_{11}) : \text{l.c.} \to \text{u.c.};\ \nu \to 1;\ T \to 0] \tag{2.63}$$

means that $Q(k_c, \Phi_{11})$ is given by taking the expression for $q(k_c, \phi_{11})$ and turning the lower case dependent variables (e.g. ϕ) into upper case dependent variables (e.g. Φ), setting $\nu = 1$ and $T = 0$. The shorthand will also be used to simplify interface conditions. Thus

$$\phi'' + k_c^2\phi = \mu\{\text{l.h.s.} : \text{l.c.} \to \text{u.c.}\}, \tag{2.64}$$

where l.h.s. means 'left-hand side', is equivalent to the third interface condition in (2.61). Before proceeding we given the following definitions:

$$\left.\begin{aligned}
\zeta_1 &= -\psi_1(0)/\sigma_r = -\Psi_1(0)/\sigma_r;\\
u_{11} &= \psi_1'(0) + \zeta_1 u_b'(0)\\
\text{and} \qquad U_{11} &= (u_{11} : \text{l.c.} \to \text{u.c.});\\
ik_c\rho p_{11}' &= (k_c^2/\nu R_c)\,[\psi_1''(0) - k_c^2\psi_1(0)] - ik_c^3\sigma_r\psi_1(0)\\
\text{and} \qquad ik_c P_{11}' &= (ik_c\rho p_{11}' : \text{l.c.} \to \text{u.c.};\ \nu \to 1).
\end{aligned}\right\} \tag{2.65}$$

The distortion of the mean flow is governed by the system occurring at $O(\epsilon E^0)$, which is

$$\left.\begin{aligned}
&\Phi_{02} = \Phi_{02}' = 0 \quad \text{on} \quad y = 1,\\
&R_c^{-1}\Phi_{02}^{(\mathrm{iv})} = \mathrm{i}k_c|A|^2(\Psi_1'\tilde{\Psi}_1 - \Psi_1\tilde{\Psi}_1')'',\\
&\left.\begin{aligned}
&\phi_{02} + \psi_1'\tilde{\zeta}_1|A|^2 = (\text{l.h.s.}:\text{l.c.}\to\text{u.c.}),\\
&\phi_{02}'' - \frac{u_b'\phi_{02}}{u_b+a_1} + |A|^2\left\{u_b''|\zeta_1|^2 - \frac{u_b'}{u_b+a_1}(\tilde{u}_{11}\zeta_1 + \psi_1'\tilde{\zeta}_1) + [\zeta_1\tilde{\psi}_1 + \text{c.c.}]\right\}\\
&\qquad = (\text{l.h.s.}:\text{l.c.}\to\text{u.c.}),\\
&\phi_{02}'' - \nu R_c \mathrm{i}k_c|A|^2(\tilde{\psi}_1\psi_1' - \psi_1\tilde{\psi}_1') = \mu(\text{l.h.s.}:\text{l.c.}\to\text{u.c.};\ \nu\to 1),\\
&(\nu R_c)^{-1}\phi_{02}''' - \mathrm{i}k_c|A|^2(\tilde{\psi}_1\psi_1' - \psi_1\tilde{\psi}_1')' = \rho(\text{l.h.s.}:\text{l.c.}\to\text{u.c.};\ \nu\to 1),
\end{aligned}\right]\ \text{on}\quad y = 0\\
&(\nu R_c)^{-1}\phi_{02}^{(\mathrm{iv})} = \mathrm{i}k_c|A|^2(\psi_1'\tilde{\psi}_1 - \psi_1\tilde{\psi}_1')'',\\
&\phi_{02} = \phi_{02}' = 0 \quad \text{on} \quad y = -d^{-1},
\end{aligned}\right\} \tag{2.66}$$

where a tilde denotes the complex conjugate. Care must be taken when solving (2.66), as we see that the homogeneous form of (2.66) is identical to the eigenvalue problem (2.21) for long waves, with $-a_1$ taking the place of c_0. Thus when the group velocity of the disturbance is not equal to the phase speed of long waves, (2.66) has a unique solution of the form

$$\left.\begin{aligned}
\Phi_{02} &= F(y)\,|A|^2,\\
\phi_{02} &= f(y)\,|A|^2,\\
\text{where}\quad f(y) &= \int_{-d^{-1}}^{y} s(y)\,\mathrm{d}y + (y+d^{-1})^2\,(by+c)\\
\text{with}\quad s(y) &= \mathrm{i}k_c\,\nu R_c\int_{-d^{-1}}^{y}(\psi_1'\tilde{\psi}_1 - \psi_1\tilde{\psi}_1')\,\mathrm{d}y,\\
\text{and}\quad F(y) &= \int_1^y S(y)\,\mathrm{d}y + (y-1)^2\,(By+C)\\
\text{with}\quad S(y) &= \mathrm{i}k_c R_c\int_1^y(\Psi_1'\tilde{\Psi}_1 - \Psi_1\tilde{\Psi}_1')\,\mathrm{d}y.
\end{aligned}\right\} \tag{2.67}$$

The constants B, C, b and c are found by enforcing the interface conditions.

At $O(\epsilon E^2)$ we obtain the system governing the second harmonic, namely

$$\left.\begin{aligned}
&\Phi_{22} = \Phi_{22}' = 0 \quad \text{on} \quad y = 1,\\
&L(2k_c,\,\Phi_{22}) = \mathrm{i}k_c A^2(\Psi_1\Psi_1''' - \Psi_1'\Psi_1''),\\
&\left.\begin{aligned}
&\phi_{22} + \tfrac{1}{2}\zeta_1\psi_1' A^2 = (\text{l.h.s.}:\text{l.c.}\to\text{u.c.}),\\
&\phi_{22}' - u_b'\phi_{22}/\sigma_r + A^2[\zeta_1\psi_1'' - \tfrac{1}{2}u_b'\zeta_1(u_{11}+\psi_1') + \tfrac{1}{2}u_b''\zeta_1^2] = (\text{l.h.s.}:\text{l.c.}\to\text{u.c}),\\
&\phi_{22}'' + 4k_c^2\phi_{22} + A^2\zeta_1[\psi_1'(8k_c^2 + \nu R_c\mathrm{i}k_c\sigma_r) - \nu R_c\mathrm{i}k_c\psi_1(u_b' + (F+k_c^2T)/\sigma_r R_c^2)]\\
&\qquad = \mu(\text{l.h.s.}:\text{l.c.}\to\text{u.c.};\ \nu\to 1;\ T = 0),\\
&q(2k_c,\ \phi_{22}) + A^2\Bigg[\mathrm{i}k_c(\psi_1\psi_1'' - \psi_1'\psi_1') + \frac{\mathrm{i}k_c\zeta_1(u_{11}+\psi_1')\,(F+4k_c^2T)}{\sigma_r R_c^2}\\
&\qquad + \zeta_1\left(2\mathrm{i}k_c\rho\boldsymbol{p}_{11}' - \frac{4k_c^2\psi''}{\nu R_c}\right)\Bigg]\\
&\qquad = \rho[\text{l.h.s.}:(\mathrm{i}k_c\rho\boldsymbol{p}_{11}')\to(\mathrm{i}k_c\boldsymbol{P}_{11}');\ \text{l.c.}\to\text{u.c.};\ \nu\to 1;\ T\to 0]
\end{aligned}\right]\ \text{on}\quad y = 0\\
&l(2k_c,\,\phi_{22}) = \mathrm{i}k_c A^2(\psi_1\psi_1''' - \psi_1'\psi_1''),\\
&\phi_{22} = \phi_{22}' = 0 \quad \text{on} \quad y = -d^{-1}.
\end{aligned}\right\} \tag{2.68}$$

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The solution to (2.68) can be written as

$$\left.\begin{aligned}\Phi_{22} &= A^2\Psi_2,\\ \phi_{22} &= A^2\psi_2,\end{aligned}\right\} \tag{2.69}$$

where we have assumed that the solution is unique.

The $O(\epsilon E)$ system is essentially the non-homogeneous system (B 5) obtained at $O(k-k_c)$ when defining a_1. As a_1 is chosen so that (B 5) has a solution, then a solution of the $O(\epsilon E)$ system exists for Φ_{12} and ϕ_{12}. The modifications to (B 5) needed to obtain the $O(\epsilon E)$ system are: Φ_{10} and ϕ_{10} are replaced by Φ_{12} and ϕ_{12} while all the non-homogeneous terms are multiplied by $(-\mathrm{i}\,\partial A/\partial\xi)$ and Φ_1 and ϕ_1 are replaced by the normalized eigenfunctions Ψ_1 and ψ_1. Thus the solution of the $O(\epsilon E)$ system is

$$\left.\begin{aligned}\Phi_{12} &= -\mathrm{i}(\partial A/\partial\xi)\Psi_{10}+A_2\Psi_1,\\ \phi_{12} &= -\mathrm{i}(\partial A/\partial\xi)\psi_{10}+A_2\psi_1,\end{aligned}\right\} \tag{2.70}$$

where Ψ_{10} and ψ_{10} are solutions of (B 5) when the forcing terms contain the normalized eigenfunctions and A_2 is an arbitrary function of τ and ξ.

Finally, at $O(\epsilon^{\frac{3}{2}}E)$, we have the system of equations

$$\left.\begin{array}{l}
\Phi_{13} = \Phi_{13}' = 0 \quad \text{on} \quad y = 1,\\[4pt]
L(k_c,\Phi_{13}) = \left(a_2\dfrac{\partial^2 A}{\partial\xi^2}-\dfrac{\partial A}{\partial\tau}\right)(\Psi_1''-k_c^2\Psi_1)-\dfrac{A(\Psi_1^{(\mathrm{iv})}-2k_c^2\Psi_1+k_c^4\Psi_1)}{d_{1\mathrm{r}}R_c^2}\\[4pt]
\qquad -\mathrm{i}\dfrac{\partial A_2}{\partial\xi}\{\cdot\}-\dfrac{\partial^2 A}{\partial\xi^2}\{\cdot\}+A|A|^2G(y),\\[4pt]
\left.\begin{array}{l}
\phi_{13}-A|A|^2 v = \Phi_{13}-A|A|^2\mathscr{V},\\[4pt]
\phi_{13}'-\dfrac{u_b'\phi_{13}}{\sigma_r}+\left(a_2\dfrac{\partial^2 A}{\partial\xi^2}-\dfrac{\partial A}{\partial\tau}\right)\dfrac{\mathrm{i}u_b'\psi_1}{k_c\sigma_r^2}\\[4pt]
\qquad -\mathrm{i}\dfrac{\partial A_2}{\partial\xi}\{\cdot\}-\dfrac{\partial^2 A}{\partial\xi^2}\{\cdot\}+A|A|^2\left[u+\dfrac{u_b'}{\sigma_r}(K_1)\right]\\[4pt]
\qquad = (\text{l.h.s.}: K_1\to K_u;\ \text{l.c.}\to\text{u.c.};\ u\to\mathscr{U}),\\[4pt]
\phi_{13}''+k_c^2\phi_{13}-\mathrm{i}\dfrac{\partial A_2}{\partial\xi}\{\cdot\}-\dfrac{\partial^2 A}{\partial\xi^2}\{\cdot\}+A|A|^2 s = \mu(\text{l.h.s.}: s\to S;\ \text{l.c.}\to\text{u.c.}),\\[4pt]
q(k_c,\phi_{13})+\left(a_2\dfrac{\partial^2 A}{\partial\xi^2}-\dfrac{\partial A}{\partial\tau}\right)\left[\psi_1'+\psi_1\dfrac{(F+k_c^2T)}{\sigma_r^2R_c^3}\right]-\dfrac{\partial^2 A}{\partial\xi^2}\{\cdot\}-\mathrm{i}\dfrac{\partial A_2}{\partial\xi}\{\cdot\}\\[4pt]
\qquad +\dfrac{A\mathrm{i}k_c}{d_{1\mathrm{r}}R_c}\left(u_b'\psi_1-\sigma_r\psi_1'-\psi_1\dfrac{F+k_c^2T}{\sigma_rR_c^2}\right)+nA|A|^2\\[4pt]
\qquad = \rho(\text{l.h.s.}: n\to\mathscr{N};\ lc\to UC;\ T\to 0),
\end{array}\right]\ \text{on}\quad y=0\\[4pt]
l(k_c,\phi_{13}) = \left(a_2\dfrac{\partial^2 A}{\partial\xi^2}-\dfrac{\partial A}{\partial\tau}\right)(\psi_1''-k_c^2\psi_1)-\dfrac{A(\psi_1^{(\mathrm{iv})}-2k_c^2\psi_1''+k_c^4\psi_1)}{d_{1\mathrm{r}}\nu R_c^2}\\[4pt]
\qquad -\mathrm{i}\dfrac{\partial A_2}{\partial\xi}\{\cdot\}-\dfrac{\partial^2 A}{\partial\xi^2}\{\cdot\}+A|A|^2g(y),\\[4pt]
\phi_{13} = \phi_{13}' = 0 \quad \text{on} \quad y = -d^{-1}.
\end{array}\right\} \tag{2.71}$$

In (2.71) the unstated coefficients $\{\cdot\}$ of $-\mathrm{i}\,\partial A_2/\partial\xi$ are given by the non-homogeneous terms in (B 5), with the normalized eigenfunction replacing ϕ_1 and Φ_1. Similarly, the unstated coefficients of $-\partial^2 A/\partial\xi^2$ are given by the non-homogeneous terms in (B 7) with ϕ_1 and Φ_1 replaced by ψ_1 and Ψ_1, respectively, and with ϕ_{10} and Φ_{10} replaced by ψ_{10} and Ψ_{10} respectively.

For (2.71) to have a solution the non-homogeneous terms must be orthogonal to the adjoint function, and thus the solvability condition leads to the amplitude equation

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1\mathrm{r}}} A + \kappa A|A|^2, \tag{2.72}$$

where a_2 and d_1 are given by (2.51) and (2.48) respectively. The coefficient of the nonlinear term is given by

$$\begin{aligned}\kappa(CDEN) = {} & \psi(\rho\mathcal{N} - n) - \psi'(\mu\mathcal{S} - s)/\nu R_\mathrm{c} \\ & + (v - \mathcal{V})\,\{[u_\mathrm{b}'(\psi'' + k_\mathrm{c}^2\psi)/\sigma_\mathrm{r} - (\psi''' - 3k_\mathrm{c}^2\psi')]/\nu R_\mathrm{c} + \mathrm{i}k_\mathrm{c}\sigma_\mathrm{r}\psi' - \mathrm{i}k_\mathrm{c}\psi(F + k_\mathrm{c}^2 T)/\sigma_\mathrm{r} R_\mathrm{c}^2\} \\ & + \left(\frac{U_\mathrm{b}' K_\mathrm{u}}{\sigma_\mathrm{r}} + \mathcal{U} - \frac{u_\mathrm{b}' K_\mathrm{l}}{\sigma_\mathrm{r}} - u\right)\frac{(\psi'' + k_\mathrm{c}^2\psi)}{\nu R_\mathrm{c}} + \int_{-d^{-1}}^{0} \psi g(y)\,\mathrm{d}y + \rho\int_0^1 \Psi G(y)\,\mathrm{d}y. \end{aligned} \tag{2.73}$$

This definition of κ has the same form as the definitions of the critical constants, with terms due to the motion in the bulk of the fluid and contributions from (nonlinear) effects at the interface. Equation (2.72) is the amplitude equation we set out to derive, and so the perturbation expansion is not continued any further; the rest of this section is devoted to defining the coefficients of the nonlinear terms in (2.71).

The coefficients of the nonlinear terms in the equations of motion are given by

$$\begin{aligned} g(y) = {} & \psi_2'(\tilde{\psi}_1'' - k_\mathrm{c}^2\tilde{\psi}_1) + 2\psi_2(\tilde{\psi}_1''' - k_\mathrm{c}^2\tilde{\psi}_1') - 2\tilde{\psi}_1'(\psi_2'' - 4k_\mathrm{c}^2\psi_2) \\ & - \tilde{\psi}_1(\psi_2''' - 4k_\mathrm{c}^2\psi_2') - f'(\psi_1'' - k_\mathrm{c}^2\psi_1) + f'''\psi_1 \end{aligned} \tag{2.74a}$$

and

$$G(y) = (g(y)\!:\!\mathrm{l.c.} \rightarrow \mathrm{u.c.}). \tag{2.74b}$$

For convenience we introduce the following quantities, where all the functions are evaluated at $y = 0$:

$$\left.\begin{aligned} \zeta_1 &= -\psi_1/\sigma_\mathrm{r} = -\Psi_1/\sigma_\mathrm{r}, \\ u_{11} &= \psi_1' + u_\mathrm{b}'\zeta_1, \\ \zeta_{02} &= -[f(0) + \tilde{u}_{11}\zeta_1 + \psi_1'\tilde{\zeta}_1]/(u_\mathrm{b} + a_1), \\ u_{02} &= f'(0) + u_\mathrm{b}'\zeta_{02} + (\zeta_1\tilde{\psi}_1'' + \mathrm{c.c.}) + u_\mathrm{b}''|\zeta_1|^2, \\ \zeta_{22} &= -[\psi_2 + \tfrac{1}{2}\zeta_1(u_{11} + \psi_1')]/\sigma_\mathrm{r}, \\ \text{and}\quad u_{22} &= \psi_2' + u_\mathrm{b}'\zeta_{22} + \zeta_1\psi_1'' + \tfrac{1}{2}u_\mathrm{b}''\zeta_1^2. \end{aligned}\right\} \tag{2.75}$$

Thus in the first interface condition, which expresses continuity of vertical velocity, we have

$$v = -\zeta_{02}\psi_1' + \zeta_{22}\tilde{\psi}_1' - 2\tilde{\zeta}_1\psi_2' - \psi_1''|\zeta_1|^2 + \tfrac{1}{2}\tilde{\psi}_1''\zeta_1^2 \tag{2.76a}$$

and

$$\mathcal{V} = (v\!:\!\mathrm{l.c.} \rightarrow \mathrm{u.c.}). \tag{2.76b}$$

From the kinematic condition we find that

$$K_\mathrm{l} = v - (u_{02}\zeta_1 - u_{22}\tilde{\zeta}_1 + 2\tilde{u}_{11}\zeta_{22}) \tag{2.77a}$$

and

$$K_\mathrm{u} = (K_\mathrm{l}\!:\!v \rightarrow \mathcal{V}). \tag{2.77b}$$

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In the interface condition expressing continuity of horizontal velocity we have

$$\mathscr{u} = \zeta_{02}\psi_1'' + f''\zeta_1 + \zeta_{22}\tilde{\psi}_1'' + \tilde{\zeta}_1\psi_2'' + \psi_1'''|\zeta_1|^2 + u_b''(\zeta_{02}\zeta_1 + \zeta_{22}\tilde{\zeta}_1) + \tfrac{1}{2}\tilde{\psi}_1'''\zeta_1^2 \qquad (2.78a)$$

and

$$\mathscr{U} = (\mathscr{u}: \text{l.c.} \rightarrow \text{u.c.}). \qquad (2.78b)$$

For the shear stress condition we introduce the following interfacial quantities:

$$\left.\begin{aligned} \mathscr{u}' &= \zeta_{02}\psi_1''' + \zeta_{22}\tilde{\psi}_1''' + \zeta_1(f''' + \psi_1^{(\mathrm{iv})}\tilde{\zeta}_1 + \tfrac{1}{2}\tilde{\psi}_1^{(\mathrm{iv})}\zeta_1) + \tilde{\zeta}_1\psi_2''';\\ \mathscr{U}' &= (\mathscr{u}': \text{l.c.} \rightarrow \text{u.c.});\\ \partial v/\partial x_{13} &= \zeta_{02}\psi_1' + \zeta_{22}\tilde{\psi}_1' + 4\tilde{\zeta}_1\psi_2' + \psi_1''|\zeta_1|^2 + \tfrac{1}{2}\tilde{\psi}_1''\zeta_1^2\\ \text{and}\quad \frac{\partial V}{\partial x_{13}} &= \left(\frac{\partial v}{\partial x_{13}}: \text{l.c.} \rightarrow \text{u.c.}\right). \end{aligned}\right\} \qquad (2.79)$$

Thus the coefficients of the nonlinear terms in the shear stress condition are

$$\mathscr{s} = \mathscr{u}' + k_c^2\,\partial v/\partial x_{13} - 4k_c^2[\zeta_1(\zeta_1\psi_1'' - \text{c.c.}) + 2\zeta_{22}\psi_1' + \tilde{\zeta}_1(2\psi_2' + \zeta_1\psi_1')] \qquad (2.80a)$$

and

$$\mathscr{S} = \left(\mathscr{s}: \mathscr{u}' \rightarrow \mathscr{U}';\ \frac{\partial v}{\partial x_{13}} \rightarrow \frac{\partial V}{\partial x_{13}};\ \text{l.c.} \rightarrow \text{u.c.}\right). \qquad (2.80b)$$

The viscous contributions to the normal stress condition requires the following quantities:

$$\left.\begin{aligned} v_{13}' &= -\zeta_{02}\psi_1'' + \zeta_{22}\tilde{\psi}_1'' - 2\tilde{\zeta}_1\psi_2'' - \psi_1''|\zeta_1|^2 + \tfrac{1}{2}\tilde{\psi}_1'''\zeta_1^2; \quad V_{13}' = (v_{13}'; \text{l.c.} \rightarrow \text{u.c.});\\ \tau_{02} &= f'' + u_b''\zeta_{02} + [\zeta_1\tilde{\psi}_1''' + \text{c.c.}] + k_c^2[\zeta_1\tilde{\psi}_1' + \text{c.c.}]; \quad T_{02} = (\tau_{02}: \text{l.c.} \rightarrow \text{u.c.});\\ \tau_{11} &= \psi_1'' + u_b''\zeta_1 + k_c^2\psi_1; \quad T_{11} = (\tau_{11}: \text{l.c.} \rightarrow \text{u.c.});\\ \tau_{22} &= \psi_2'' + u_b''\zeta_{22} + \zeta_1\psi_1''' + 4k_c^2\psi_2 + k_c^2\zeta_1\psi_1'; \quad T_{22} = (\tau_{22}; \text{l.c.} \rightarrow \text{u.c.});\\ \mathscr{v}_{13} &= 4ik_c^3\psi_1'|\zeta_1|^2 + 2ik_c^3\tilde{\psi}_1'\zeta_1^2 + ik_c(-\zeta_1\tau_{02} - 2\zeta_{22}\tilde{\tau}_{11} + \tilde{\zeta}_1\tau_{22})\\ \text{and}\quad \mathscr{V}_{13} &= (\mathscr{v}_{13}; \text{l.c.} \rightarrow \text{u.c.}). \end{aligned}\right\} \qquad (2.81)$$

The contribution of the pressure to the normal stress condition is given by

$$\left.\begin{aligned} ik_c\rho\mathscr{p}_{13} &= \zeta_{02}(ik_c\rho\boldsymbol{p}_{11}') + ik_c\zeta_1(\rho\boldsymbol{p}_{02}') - \zeta_{22}(ik_c\rho\boldsymbol{p}_{11}')\\ &\qquad + \tfrac{1}{2}\zeta_1(2ik_c\rho\boldsymbol{p}_{22}') + |\zeta_1|^2(ik_c\rho\boldsymbol{p}_{11}'') - \tfrac{1}{2}\zeta_1^2(ik_c\rho\boldsymbol{p}_{11}'')\\ \text{and}\quad ik_c\mathscr{P}_{13} &= (ik_c\rho\mathscr{p}_{13}: \text{l.c.} \rightarrow \text{u.c.};\ \rho \rightarrow 1),\\ \text{where}\ ik_c\rho\boldsymbol{p}_{11} &= (\psi_1''' - k_c^2\psi_1')/\nu R_c + ik_c(u_b'\psi_1 - \sigma_r\psi_1'),\\ ik_c\boldsymbol{P}_{11} &= (ik_c\rho\boldsymbol{p}_{11}: \text{l.c.} \rightarrow \text{u.c.};\ \nu \rightarrow 1), \quad ik_c\rho\boldsymbol{p}_{11}' = k_c^2(\psi_1'' - k_c^2\psi_1)/\nu R_c - ik_c^3\sigma_r\psi_1,\\ ik_c\boldsymbol{P}_{11}' &= (ik_c\rho\boldsymbol{p}_{11}': \text{l.c.} \rightarrow \text{u.c.};\ \nu \rightarrow 1), \quad ik_c\rho\boldsymbol{p}_{11}'' = k_c^2(ik_c\rho\boldsymbol{p}_{11} - 2ik_c u_b'\psi_1),\\ ik_c\boldsymbol{P}_{11}'' &= (ik_c\rho\boldsymbol{p}_{11}'': \text{l.c.} \rightarrow \text{u.c.}), \quad \rho\boldsymbol{p}_{02}' = -2k_c^2(\psi_1'\tilde{\psi}_1 + \text{c.c.}),\\ \boldsymbol{P}_{02}' &= (\rho\boldsymbol{p}_{02}': \text{l.c.} \rightarrow \text{u.c.}), \quad 2ik_c\rho\boldsymbol{p}_{22}' = 4k_c^2(\psi_2'' - 4k_c^2\psi_2)/\nu R_c - 8ik_c^3\sigma_r\psi_2\\ \text{and}\quad 2ik_c\boldsymbol{P}_{22}' &= (2ik_c\rho\boldsymbol{p}_{22}': \text{l.c.} \rightarrow \text{u.c.};\ \nu \rightarrow 1). \end{aligned}\right\} \qquad (2.82)$$

Thus by using (2.77), (2.81) and (2.82) the coefficients of the nonlinear terms in the pressure condition are given by

$$\begin{aligned} \mathscr{n} = {} & ik_c(-f'\psi_1' - \tilde{\psi}_1'\psi_2' + f''\psi_1 + 2\psi_2\tilde{\psi}_1'' - \psi_2''\tilde{\psi}_1) - ik_c(K_1)\,(F + k_c^2 T)/\sigma_r R_c^2 + 2k_c^2 v_{13}'/\nu R_c\\ & + ik_c\rho\mathscr{p}_{13} - 2ik_c\mathscr{v}_{13}/\nu R_c + 3Tik_c^5\zeta_1|\zeta_1|^2/2R_c^2 \end{aligned} \qquad (2.83a)$$

and

$$\mathscr{N} = (\mathscr{n}: K_1 \rightarrow K_{\hat{u}};\ ik_c\rho\mathscr{p}_{13} \rightarrow ik_c\mathscr{P}_{13};\ v_{13}' \rightarrow V_{13}';\ \mathscr{v}_{13} \rightarrow \mathscr{V}_{13};\ \nu \rightarrow 1;\ T \rightarrow 0;\ lc \rightarrow UC). \qquad (2.83b)$$

3. Results and discussion

The analysis of the preceding sections has shown that the amplitude of disturbances to the basic flow is governed by

$$\frac{\partial A}{\partial \tau} - a_2 \frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}} A + \kappa A|A|^2, \; R > R_c, \tag{3.1}$$

where the constant coefficients are defined in §2.3. The main object of this work has been to obtain numerical values for these constants. For a given basic flow this has involved the accurate determination of the critical conditions for linear stability, and then the numerical solution of the systems (2.35), (B 5), (2.61), (2.66) and (2.68). In developing the programs to perform

Table 7. Constants for nonlinear analysis from p.P.f. profile

k_s	R_c	k_c	c_r	a_{1r}	d_1	a_2	κ
3.6	380.6	1.818	0.711	−0.667	(0.895*E*−5, 0.217*E*−2)	(0.136*E*−2, −0.157*E*−2)	(−29.7, 237.9)
7.3	445.3	2.543	1.17	−0.896	(0.764*E*−5, 0.513*E*−2)	(0.134*E*−1, −0.273*E*−1)	(−1.95, −3271)
14.6	789.7	3.750	1.45	−0.973	(0.200*E*−4, 0.552*E*−2)	(0.317*E*−2, −0.240*E*−1)	(1.50, −5581)
29.2	1835	4.043	1.62	−1.01	(0.364*E*−5, 0.294*E*−2)	(0.111*E*−2, −0.384*E*−1)	(7.9, −6053)
58.4	4263	4.627	1.80	−1.1	(0.12*E*−5, 0.16*E*−2)	(0.50*E*−3, −0.40*E*−1)	(−23 −7973)

Computations were performed on a uniform mesh with a step length of 1/450.

Table 8. Constants for nonlinear analysis from b.l.1 profile

k_s	R_c	k_c	c_r	a_{1r}	d_1	a_2	κ
3.6	589.9	1.416	0.406	−0.367	(0.404*E*−5, 0.742*E*−3)	(0.557*E*−3, −0.170*E*−1)	(−9.95, 912.3)
7.3	1341	1.750	0.443	−0.350	(0.110*E*−5, 0.446*E*−3)	(0.337*E*−3, −0.327*E*−1)	(−8.15, 827.1)
14.6	1142	1.959	1.17	−0.762	(0.223*E*−5, 0.183*E*−2)	(0.739*E*−3, −0.115*E*0)	(−3.82, −335.9)
29.2	2330	2.250	1.49	−0.898	(0.837*E*−6, 0.134*E*−2)	(0.526*E*−3, −0.125*E*0)	(−1.5, −479)
58.4	5173	2.643	1.74	−0.99	(0.30*E*−6, 0.84*E*−3)	(0.29*E*−3, −0.11*E*0)	(−0.96, −792)
116.7	12276	2.944	1.96	−1.1	(0.92*E*−7, 0.44*E*−3)	(0.13*E*−3, −0.10*E*0)	(−7.1, −1042)

Computations were performed on a uniform mesh with a step length of 1/450, except for k_s = 58.4 and 116.7 where the step length was 1/800.

Table 9. Constants for nonlinear analysis from b.l.2 profile

k_s	R_c	k_c	c_r	a_{1r}	d_1	a_2	κ
3.6	269.8	0.8746	1.18	−0.958	(0.554*E*−5, 0.275*E*−2)	(0.165*E*−2, −0.159*E*0)	(−0.221, 20.61)
7.3	293.9	1.365	2.15	−1.41	(0.196*E*−4, 0.835*E*−2)	(0.209*E*−2, −0.187*E*0)	(−1.02, 305.3)
14.6	674.7	1.544	2.42	−1.49	(0.453*E*−5, 0.472*E*−2)	(0.104*E*−2, −0.198*E*0)	(−0.81, 923.8)
29.2	1582	1.844	2.64	−1.56	(0.148*E*−5, 0.265*E*−2)	(0.470*E*−3, −0.170*E*0)	(−4.1, 1137)
58.4	3852	2.201	2.79	−1.6	(0.46*E*−6, 0.14*E*−2)	(0.20*E*−3, −0.14*E*0)	(0.8, −8652)

(*a*) For the air: for k_s = 3.6, 7.3 and 14.6 the computations were performed with a step length of 1/450; for k_s = 29.2 a step length of 1/800 was used and for k_s = 58.4 a step length of 1/1000 was used.
(*b*) For the water: the step length was twice the step length for the air.

Table 10. Constants for nonlinear analysis from p.C.f. profile

k_s	R_c	k_c	c_r	a_{1r}	d_1	a_2	κ
3.65	849.4	1.500	2.56	−0.224	(0.222*E*−5, 0.376*E*−3)	(0.420*E*−3, −0.110*E*−1)	(−19.2, 1142)
14.6	5650	2.280	2.50	−0.174	(0.147*E*−6, 0.817*E*−4)	(0.942*E*−4, −0.153*E*−1)	(−16.5, 995)

Computations were performed on a uniform mesh with a step length of 1/450.

these calculations care was taken to compare our results with independent calculations on problems that are special cases of the flows studied here. In this way the integral contributions to all the coefficients were checked. As a_1, a_2 and d_1 are properties of the linear stability problem, numerical values for these constants obtained from the definitions in §2.3.2 could be checked by numerical differentiation about the critical conditions. Further details of these checks are given in appendix A.

One part of the calculation that could not be checked by comparison with published work was the contribution of the interface terms to the value of κ. Here there are two possible sources of error: algebraic mistakes in the derivation of the interface conditions and errors in programming the relevant systems. The former category of error was eliminated by several independent manual derivations of the interface conditions and finally a computer algebra program was written to derive them. The results of the computer algebra program confirmed the manually obtained results and thus we are confident that the interface conditions and definition of κ are free of algebraic errors. Errors in coding the systems of equations are harder to eliminate. Here, repeated programming of each interface condition was used: each time a slightly different algebraic form for the condition was used and checked to make sure that the results had not changed. Thus we are also confident that the numerical values of κ given by the program suffer only from the truncation error inherent in the numerical solution of the equations. Tables 7–10 give the results for the profiles studied for a range of values of k_s; the values of a_1, d_1, a_2 and κ are believed to be accurate to the number of figures quoted.

From the results we see that in general R_c increased as k_s increased and hence the eigenfunction tended to develop thin viscous layers (see figure 8). This necessitated the use of a very fine mesh to calculate the critical conditions accurately. The fine mesh adequately resolved the viscous layers but was very inefficient over the rest of the flow domain and eventually lead to storage problems in the computer which limited the range of k_s that could be studied. Clearly a variable grid or adaptive method of solving the governing equations would be an improvement on the $O(h^4)$ uniform mesh technique used here.

The accuracy of the results was checked by repeating several typical calculations on finer meshes so that the number of significant figures in the results could be estimated. This technique worked well for the smaller values of k_s, but at the larger values of k_s storage limitations prevented any significant refinement of mesh and hence the accuracy of the results at large k_s is less certain.

An unexpected source of error arose during the calculation of κ. The numerical computations were performed so that the contribution of the bulk of the fluid (i.e. the integral terms in 2.73) to the value of κ could be isolated from the interface effects. In almost all situations the integrals gave a positive contribution to κ_r, as is to be expected from the results for plane Poiseuille flow, while the interface terms gave a negative contribution to κ_r of the same order of magnitude as the integral terms. Thus, κ_r is the difference of two nearly equal quantities and consequently many significant figures in κ_r were lost.

For the profiles b.l.1, b.l.2 and p.C.f. we see that κ_r is almost always negative, and hence nonlinear effects tend to decrease the rate of wave growth. Also, when $\kappa_r < 0$ equilibrium amplitude solutions of (3.1) are possible. For the p.P.f. profile κ_r takes both positive and negative values and hence nonlinear effects either increase or decrease the growth rate. However, as this profile has little similarity to experimental or oceanic wind wave generation we shall not discuss it further.

For $\kappa_r < 0$ the unmodulated, i.e. independent of ξ, equilibrium amplitude solution of 3.1 is

$$A = \frac{1}{(-\kappa_r)^{\frac{1}{2}}} \exp\left[i\tau\left(\frac{d_{1i}}{d_{1r}} + \frac{\kappa_i}{(-\kappa_r)}\right)\right]. \tag{3.2}$$

Here we see that as well as providing the means of establishing an equilibrium amplitude solution, the nonlinear effects have also changed the frequency of the travelling wave *via* the $\kappa_i/(-\kappa_r)$ term in (3.2). The d_{1i}/d_{1r} term represents a *linear* effect and gives the change in frequency of the disturbance due to the increase of R above R_c. The importance of nonlinearity can be seen more clearly by examining the interfacial displacement of the equilibrium wind waves and comparing it to the surface displacement of a classical nonlinear Stokes wave-train. Such a comparison has been made experimentally by Lake & Yuen (1978), while we are in a position to make theoretical comparisons.

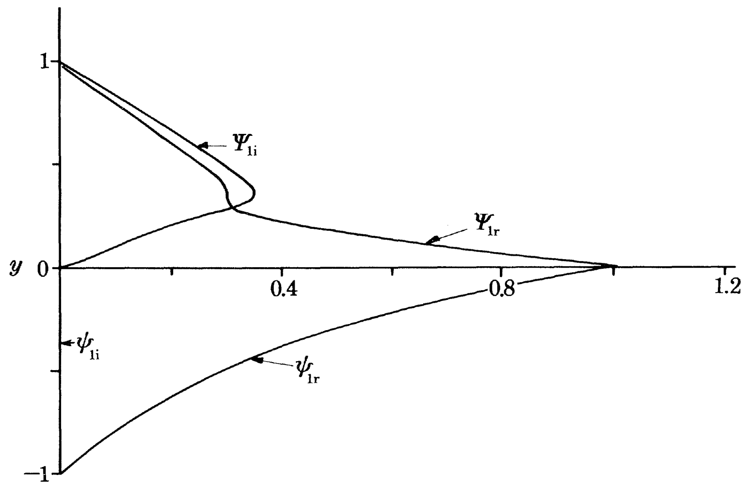

FIGURE 8 (*a*). Eigenfunction at critical conditions for b.l.1 profile with $k_s = 29.2$.

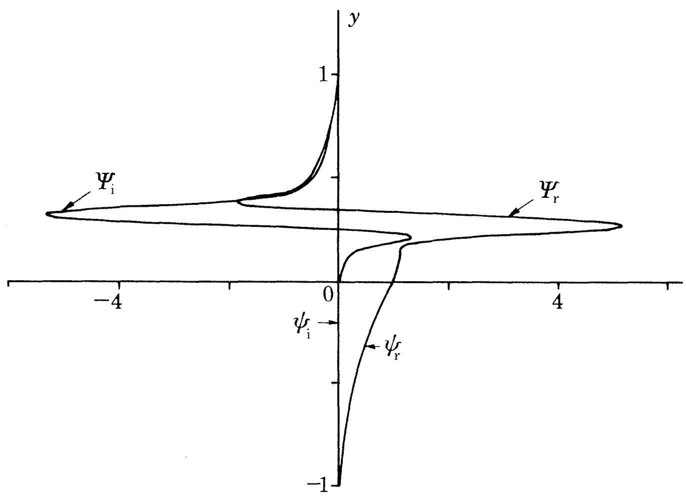

FIGURE 8 (*b*). Adjoint function at critical conditions for b.l.1 profile with $k_s = 29.2$.

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Lake & Yuen (1978) made experimental comparisons between wind-driven waves (under steady wind conditions and at fixed fetch) and mechanically generated nonlinear Stokes waves, and found many similarities in energy spectra, zero-crossing frequency and other properties of the two wave trains. This lead them to conclude that wind waves behave as a nonlinear wave train; our analysis strengthens this view as the equation governing the growth of disturbances, (3.1), is similar to the nonlinear Schrodinger equation which governs Stokes waves. However, experimentally generated wind waves contain frequency components not occurring in our calculations, and so it is not possible to use our results to support the conclusion that wind waves, in practice, form a non-dispersive, nonlinear wave train.

From (2.58) and (2.65) the leading-order surface displacement is

$$\eta = \epsilon^{\frac{1}{2}}[(-A/\sigma_r)\,E + \text{c.c.}],$$

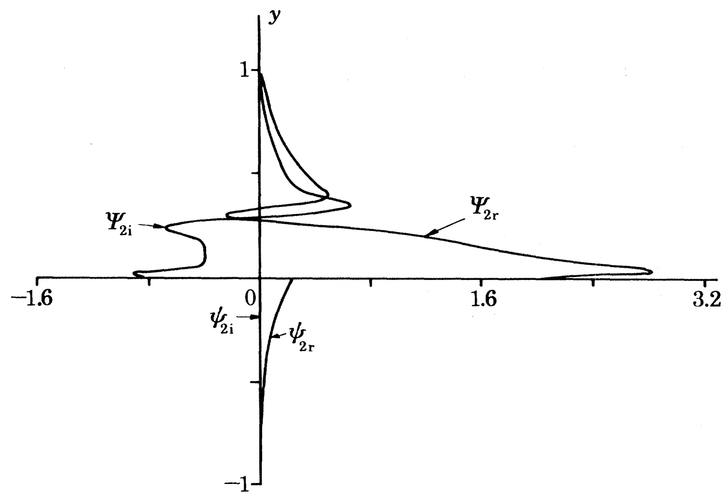

FIGURE 8 (c). Second harmonic at critical conditions for b.l.1 profile with $k_s = 29.2$.

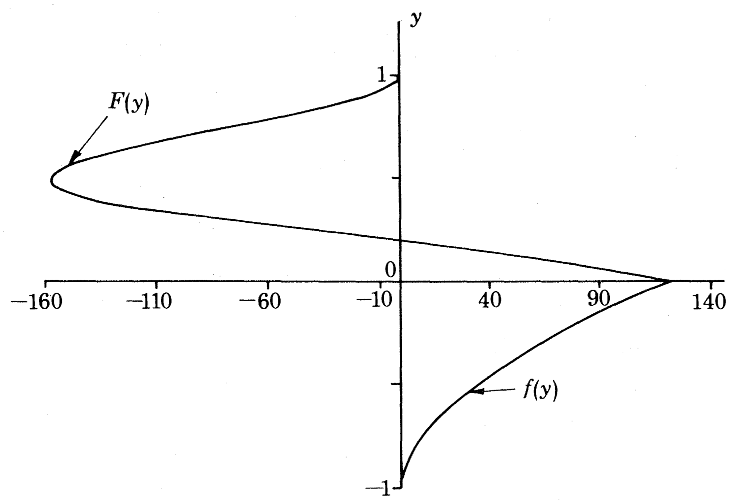

FIGURE 8 (d). Distortion of the mean flow at critical conditions for profile b.l.1 with $k_s = 29.2$.

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which becomes

$$\eta = -\frac{2\epsilon^{\frac{1}{2}}}{(-\kappa_r)^{\frac{1}{2}}\,\sigma_r}\cos\left[k_c(x-c_r t)+\frac{\kappa_i}{(-\kappa_r)}\,\epsilon t+\frac{d_{1i}}{d_{1r}}\,\epsilon t\right] \tag{3.3}$$

when (3.2) is used to eliminate A. To compare (3.3) with the nonlinear Stokes wave (on infinitely deep water) we introduce the quantities

$$a = -\frac{2\epsilon^{\frac{1}{2}}}{\sigma_r(-\kappa_r)^{\frac{1}{2}}},\quad \alpha = -\frac{\kappa_i\,\sigma_r^2}{2k_c^3 c_r},\quad \text{and}\quad \beta = \frac{d_{1i}\,\kappa_r\,\sigma_r^2}{d_{1r}\,4k_c^3 c_r}. \tag{3.4}$$

The equilibrium amplitude wind wave given by (3.3) is now

$$\eta_w = a\cos\left[k_c x - k_c c_r t(1+\tfrac{1}{2}\alpha(ak_c)^2+\beta(ak_c)^2)\right]. \tag{3.5}$$

Values for α and β for the b.l.1 and b.l.2 profiles when $\kappa_r < 0$ are given in table 11. We note that the change in frequency due to linear effects is given by the constant β, so that when comparing (3.5) with finite amplitude Stokes waves, attention will be focused on the value of α.

Table 11. Properties of equilibrium amplitude solutions

b.l.1				b.l.2		
α	β	stability	k_s	α	β	stability
−30.91	−30.9	stable	3.6	−7.95	−21.2	stable
−17.45	−34.9	stable	7.3	−85.6	−60.9	stable
20.8	−97.0	unstable	14.6	−211	−96.7	stable
26.2	−66.6	unstable	29.2	−172	−553	stable
32.1	−54.7	unstable	58.4			
35.0	−574	unstable	116.7			

The leading-order surface displacement in a Stokes wave is

$$\eta_s = a\cos\left[kx - kct(1+\tfrac{1}{2}(ak)^2)\right], \tag{3.6}$$

where, for infinitesimal waves on an infinitely deep inviscid fluid, the phase speed c is given by

$$c^2 = g/k.$$

As noted in the discussion of the linear stability results, c_r is not given by the dispersion relation for infinitesimal, inviscid surface waves. However, here we are interested in comparing the magnitude of the *change* in the two phase speeds due to nonlinear effects.

From table 11 we see that $\alpha \approx O(20)$ for b.l.1 and $\alpha \approx O(100)$ for b.l.2, and hence that the change in phase speed of a wind wave is much greater than that occurring in a Stokes wave of the same wavenumber and amplitude. (Note that the change in phase speed is $\frac{1}{2}\alpha(ak)^2$, so that despite the large values of α, the nonlinear effects are still small perturbations in the limit $ak \to 0$.) This relatively large effect is slightly surprising as it is usually assumed that, owing to the small value of ρ_{air}/ρ_{water}, the motion of the air and water can be decoupled and hence the air will have only a small effect on the properties of the wave. Also, in the definition of κ, (2.73), the integral contribution from the air is multiplied by ρ, and hence we might expect this term to be much smaller than the integral contribution from the water. That these expectations are unfounded can easily be seen by examining the magnitudes of both the integrands, as defined by (2.74), and recalling that, owing to the presence of the critical layer, the largest velocities and velocity gradients occur in the air. Figure 8 shows the various functions involved for the case of b.l.1 and $k_s = 29.2$, and it is clear that the magnitude of the nonlinear terms will be much larger for the air than for the water. Thus, from these results we conclude that for a nonlinear analysis of wave growth it is necessary to treat the air and water motion as a coupled problem.

Our results do not provide a definitive statement on the direction of the change in phase speed, as for b.l.1 α changes sign, while for b.l.2 α is always negative. Positive values of α indicate that nonlinear effects lead to increased phase speed, as is the case for Stokes waves on an infinitely deep fluid, while negative α means finite amplitude waves travel more slowly than infinitesimal waves. The negative values for α are not surprising when we compare our results to those for waves on a finite depth of inviscid fluid. For this latter case it is known (see Whitham 1974, p. 562) that when the (dimensional) wavenumber multiplied by the (dimensional) depth of the fluid is less than 1.36, finite amplitude waves travel more slowly than infinitesimal waves. In our non-dimensional notation this condition becomes

$$k/d < 1.36, \text{ nonlinearity reduces phase speed,}$$

and

$$k/d > 1.36, \text{ nonlinearity increases phase speed.}$$

In b.l.1 α changes sign when $k/d \approx 1.8$, while in b.l.2 α is negative for values of k/d up to 3.6. These results agree qualitatively with the results for waves on an inviscid fluid, and considering that our analysis has assumed a different basic flow and included nonlinear processes in the air, the agreement is seen to be quite good.

Closely associated with the effect of nonlinearity on the phase speed is the question of the stability of these equilibrium amplitude solutions. For waves on a finite depth of inviscid fluid it is known that unmodulated finite amplitude waves are stable for $k/d < 1.36$ and unstable for $k/d > 1.36$ (Whitham 1974, p. 562). The stability of unmodulated equilibrium amplitude solutions of (3.1) has been examined by Stuart & Di Prima (1978). When their stability criteria are applied to our situation, (case I of table 2 in Stuart & Di Prima applies) results analogous to the case of inviscid waves are obtained. For b.l.2 where $\alpha < 0$ for all k_s (with $\kappa_r > 0$), the waves are stable, while in b.l.1 they are stable for $\alpha < 0$ and unstable for $\alpha > 0$. So far we have discussed only those conditions for which $\kappa_r < 0$. Positive values for κ_r, i.e. increased growth rate with increasing amplitude of the disturbance, were obtained in the p.P.f. profile for $k_s = 14.6$ and $k_s = 29.2$ and for b.l.2 with $k_s = 58.4$. We are reasonably confident that the positive κ_r for the p.P.f. profile are correct. However, this profile is not really relevant to wind generation of waves in natural conditions. For the b.l.2 profile there is less certainty about the positive value for κ_r as these calculations were quite difficult to check. It is possible that given better resolution of the interfacial layers, the positive contribution to κ_r from the bulk of the fluid would be overcome by the negative contribution to κ_r from the interfacial effects. Clearly more calculations need to be performed for the b.l. profiles to see if κ_r is always negative.

Finally, we conclude the report with a brief summary of our main results.

(*a*) A linear instability mechanism can generate surface waves in the laminar flow of air over water, i.e. turbulence is not necessary for the generation of surface waves.

(*b*) The waves first produced by the instability mechanism are much longer than the waves usually generated experimentally, and they are initiated at much lower friction velocities than the experimentally observed waves.

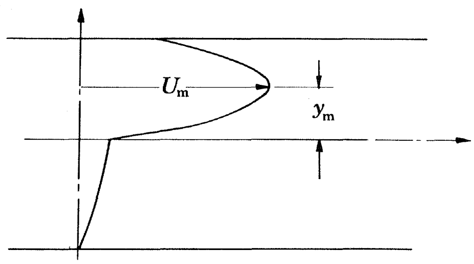
(*c*) The growth of finite amplitude wind-waves is governed by a nonlinear equation similar to the nonlinear Schrödinger equation, and hence wind-waves and Stokes waves have several qualitative similarities.

(*d*) In the velocity profiles used to model a boundary layer flow of air over water, the numerical results indicated that nonlinear effects reduce the growth rate of surface waves, and hence give rise to equilibrium amplitude waves.

(*e*) The effect of nonlinearity on the phase speed of the wind-waves was much greater than the corresponding effect on Stokes waves. Thus it was concluded that nonlinear effects in the air can be as large, or larger than, nonlinear interactions in the water, and hence that it is not possible to decouple the motion in the air and the water when examining finite amplitude wind-waves.

TABLE 12. SOME PROPERTIES OF THE BASIC FLOW PROFILES

	p.P.f.	b.l.1	b.l.2	p.C.f.
U_p	1	0.2704	0.3453	0
$U_b(1)$	0	2.831	3.601	3.881
U_m	2.656	2.831	3.601	—
y_m	0.4846	1	1	—
$U_b(0)$	0.3071	0.1265	0.3994	0.0596
$dU_b/dy\|_{y=0}$	9.694	5.409	6.404	3.821



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Appendix A

The eigenvalue problem (2.61) and its adjoint (2.35), and the non-homogeneous system (2.68) and (B 5) are special cases of the general fourth-order boundary value problem

$$\left.\begin{aligned}
\Phi = \Phi' &= 0 \quad \text{on} \quad y = 1,\\
\Phi^{(\mathrm{iv})} + C(y)\,\Phi'' + D(y)\,\Phi + 2E(y)\,\Phi' &= \mathscr{F}(y), \quad 0 < y < 1,\\
\sum_{p=0}^{3} FL_{np}\,\phi^{(p)} + FLNH_n = \sum_{p=0}^{3} FU_{np}\,\Phi^{(p)} &+ FUNH_n, \quad y = 0, \quad n = 1, 2, 3, 4,\\
\phi^{(\mathrm{iv})} + c(y)\,\phi'' + d(y)\,\phi + 2e(y)\,\phi' &= f(y), \quad -d^{-1} < y < 0\\
\phi = \phi' &= 0 \quad \text{on} \quad y = -d^{-1}.
\end{aligned}\right\} \tag{A 1}$$

Thus it is sufficient to describe the numerical solution of the system (A 1).

The numerical solution to (A 1) was obtained by a finite difference scheme similar to that used by Osborne (1967). As the governing *equations* do not contain any third derivatives a Numerov transformation can be used to increase the accuracy of the solution. Thus, on a uniform mesh with step length h we introduce the auxiliary functions G and g such that

$$\left.\begin{aligned}
\Phi &= (1 + \tfrac{1}{6}\delta^2 - \tfrac{1}{720}\delta^4)\,G \quad \text{for the upper fluid,}\\
\text{and} \qquad \phi &= (1 + \tfrac{1}{6}\delta^2 - \tfrac{1}{720}\delta^4)\,g \quad \text{for the lower fluid.}
\end{aligned}\right\} \tag{A 2}$$

Here δ is the usual central difference operator. In terms of the auxiliary functions we obtain the following approximate formulae for the derivatives of ϕ:

$$\left.\begin{aligned}
h^4\phi^{(\mathrm{iv})} &= \delta^4 g + O(\delta^{10}),\\
h^2\phi'' &= (\delta^2 + \tfrac{1}{12}\delta^4)\,g + O(\delta^6),\\
\text{and} \qquad h\phi' &= \mu\delta g + O(\delta^5).
\end{aligned}\right\} \tag{A 3}$$

The constant step size h is given by $\qquad h = y_j - y_{j-1} \qquad$ (A 4) and μ is the averaging operator. Thus our finite difference approximations have a truncation error which is $O(h^4)$ and involve the values of g at no more than five mesh points.

The interface conditions in (A 1) involve the third derivative of the solution, and so special care must be taken to maintain $O(h^4)$ accuracy. Again we follow Osborne (1967) and note that ϕ''' can be written as

$$h^3\phi''' = \mu\delta^3(1+\tfrac{1}{6}\delta^2-\tfrac{1}{720}\delta^4)^{-1}\,\phi-\tfrac{1}{12}\mu\delta(h^4\phi^{(\mathrm{iv})})+O(\delta^7).$$

Thus, using (A 2) and the appropriate equation of motion from (A 1) we obtain

$$h^3\phi''' = \mu\delta^3 g-\tfrac{1}{12}h^4\mu\delta(f-c\phi''-d\phi-2e\phi')+O(\delta^7),$$

which simplifies to

$$h^3\phi''' = \mu\delta^3 g+\tfrac{1}{12}h^2\mu\delta(c\delta^2 g+h^2dg+h2e\mu\delta g-h^2 f), \qquad \text{(A 5)}$$

and hence we have maintained $O(h^4)$ accuracy and a five point formula for the third derivative.

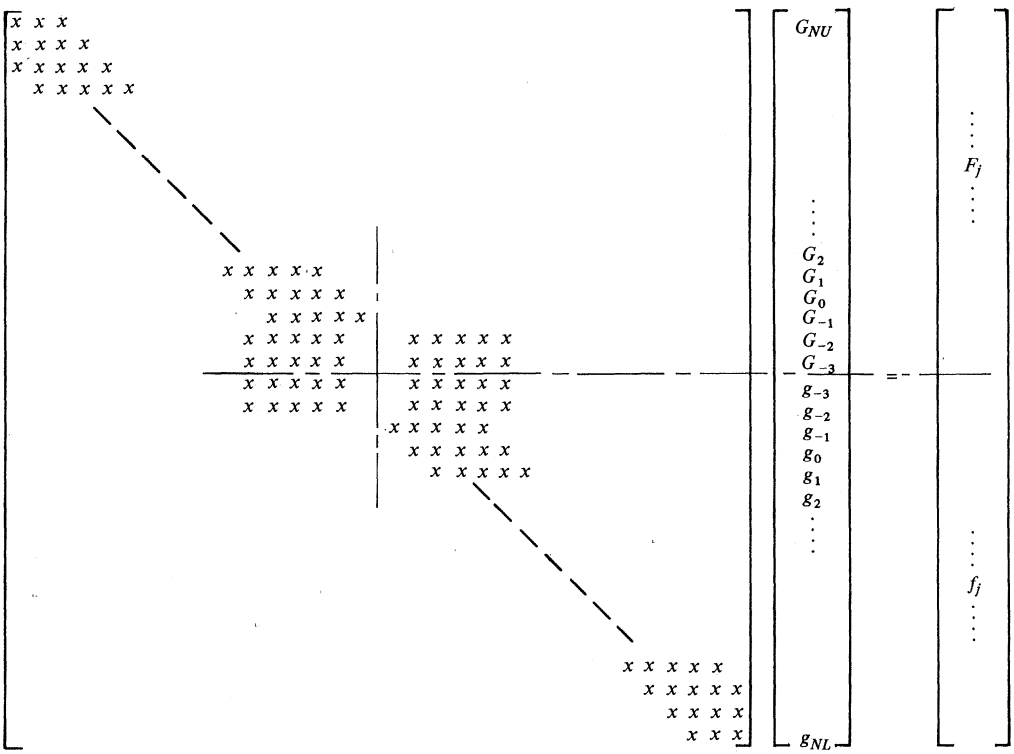

FIGURE A1. Form of the finite difference equations; x indicates a non-zero element.

The substitution of (A 3) into the governing equations in (A 1) yields the finite difference approximation to the equations:

$$\left.\begin{array}{l} \mathscr{A}_1 g_{j-2}+\mathscr{A}_2 g_{j-1}+\mathscr{A}_3 g_j+\mathscr{A}_4 g_{j+1}+\mathscr{A}_5 g_{j+2} = h^4 f_j, \\ \text{where} \quad \mathscr{A}_1 = \mathscr{A}_5 = 1+\tfrac{1}{12}h^2c_j, \quad \mathscr{A}_2 = -4+\tfrac{2}{3}h^2c_j+\tfrac{1}{6}h^4d_j-h^3e_j, \\ \mathscr{A}_4 = \mathscr{A}_2+2h^3e_j, \quad \text{and} \quad \mathscr{A}_3 = 6-\tfrac{3}{2}h^2c_j+\tfrac{2}{3}h^4d_j. \end{array}\right\} \qquad \text{(A 6)}$$

The subscript j means that the function concerned is evaluated at $y = y_j$. (Note that an analogous form of difference equation is obtained for the upper fluid.) The equation (A 6)

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involves the values of g at five mesh points centred on $y = y_j$, so that when (A 6) is to be applied near the boundary $y = -d^{-1}$ or the interface $y = 0$, the mesh must be extended out of the flow domain $(-d^{-1}, 0)$ to retain the central difference formulae (A 3). Sufficient equations to find the values of the auxiliary functions at the points outside the flow domain are provided by the boundary conditions and interface conditions in (A 1). To solve (2.61), (2.35) and (B 5) only two fictitious points on each side of the flow domain were needed. However to retain $O(h^4)$ accuracy in the $O(\epsilon E^2)$ system (2.68), it was necessary to have the value of the eigenfunction at an additional point on either side of the flow region. These extra values for the auxiliary function were found by applying (A 6) at the first mesh point outside the flow domain. Thus the resulting system of algebraic equations for the auxillary function has the form shown schematically in figure A1.

Table A1. Values of c for no basic flow

Fluid properties: air over water, see table 1.
Geometry: $d = 1$; $k_s = 3.65$.
Wavenumber: $k = 3$.

analytical solution c_0	numerical solution c	step length
(160.468, −0.516379)	(160.468, −0.516423)	1/200
	(160.468, −0.517087)	1/100
(0, −0.675288)	(0, −0.675290)	1/100
(0, −1.70226)	(0, −1.70227)	1/100

The equations were solved by using Gaussian elimination without pivoting, with the elimination commencing at the top left *and* bottom right elements of the coefficient matrix and working towards the region of the interface conditions. For the eigenvalue problems (2.61) and (2.35) the F_i and f_i are identically zero and we have a homogeneous system of equations for the auxiliary functions. The eigenvalue c was found by regarding the determinant of the coefficient matrix as an analytic function of c, and by using Muller's method to locate the values of c for which the matrix is singular. The auxiliary function was then found from back substitution, after an arbitrary normalization condition had been imposed. For the non-homogeneous system (B 5) the coefficient matrix was singular, but a solution could be found by using the value of a_1 that made the forcing terms satisfy the appropriate orthogonality condition. In the non-homogeneous system (2.68) the coefficient matrix was non-singular and a unique solution could be obtained.

The integrations required in the definitions of the constants a_1, a_2, d_1 and κ were performed by using Simpson's rule. However in the calculation of the distortion of the mean flow it was necessary to use a more elaborate integration scheme. From (2.67) it can be seen that ϕ_{02} and Φ_{02} are given by a double integral of the Reynolds stress terms. For any given value of y_i an even number of mesh points could not be guaranteed and hence Simpson's rule could not be used. Instead, cubic spline interpolation of the integrand was used and the integrals evaluated from the interpolating spline. This scheme retained the $O(h^4)$ accuracy of the calculations.

As mentioned in this paper tests were carried out during the development of computer programs to check with previously published results for special cases of our problem. The first tests carried out were those comparing the results of the analytical solution of §2.1.2 with the results of the numerical solution of (2.29). For computational purposes (2.29) could be obtained from the appropriate form of (A 1) (namely (2.19)) by setting $R = 1$ and $U_b(y) \equiv 0$. Some typical results obtained from the two methods of solution are given in table A1. We note that for the conditions given in table A1 the speed of surface waves, on the assumption that the air and water are inviscid fluids, is 160.5 cm s^{-1}, correct to four significant figures.

For the purely imaginary eigenvalues the agreement between the two methods of solution is excellent, while for the travelling wave disturbances we have good agreement between the results. We see that to calculate the damping of the wave quite a fine mesh is needed to obtain four figure accuracy. However, in the computations to find neutrally stable conditions we only need to know the order of magnitude of the damping, and so a relatively coarse mesh will suffice.

Table A2. Critical condition for plane Poiseuille flow

	from Davey, Hocking & Stewartson (1974)	numerical solution with step length of 1/200
R_c	3848.14	3848.23
k_c	1.02055	1.02057
c_r	0.396	0.396000

Table A3. The long-wave instability

	analytical solution		numerical solution		$(k = 10^{-4})$
profile	F/R^2	c_r	F/R^2	c_r	k_s
b.l.1	0.1164	0.3991	0.1165	0.3991	3.65
			0.1169	0.3983	14.2
b.l.2	0.9778	1.578	0.9782	1.577	3.65

Table A4. Constants for nonlinear analysis of plane Poiseuille flow of a homogeneous fluid

	from Davey, Hocking & Stewartson (1974)	numerical solution with step length of 1/200
a_1	-0.575	-0.575
a_2	(0.280, 0.0412)	(0.280, 0.0412)
d_1	$(0.379, 1.83) \times 10^{-5}$	$(0.379, 1.83) \times 10^{-5}$
κ	$(20.5, -115.6)$	$(20.6, -115.2)$

The next special case of (2.19) that was examined was the plane Poiseuille flow of a homogeneous fluid; this case is obtained by setting $\nu = \rho = d = 1$ and $T = 0$ in (2.19). In table A2 the results of our numerical solution are compared with the results given in Davey *et al.* (1974) (with allowance being made for the different scalings). The agreement is again good, the difference between the results being less than one unit in the fifth figure.

The final check on the program to solve (2.19) was provided by the long-wave length instability given in §2.1.1. Typical results from the analytical and numerical solutions are given in table A3, and the agreement is seen to be good.

Thus the program designed to solve the system (2.19) (or (2.61)) passed all the tests available, and so we can be reasonably confident of its accuracy. The adjoint system (2.35) is easily checked by requiring its eigenvalues to be the same as those of (2.61). The systems (B 5), (B 6) and (B 7) which define the coefficients a_1, d_1 and a_2 respectively have the same operator as the eigenvalue problem and hence only the non-homogeneous terms need be checked. The check on the correct programming of the non-homogeneous system (2.68) and the integrations involved in (2.67) was provided by the results given in Davey *et al.* (1974) for plane Poiseuille flow of a homogeneous fluid. The results are given in table A4.

Another check on the whole program (i.e. solution of (2.61), (2.35), (2.67) and (2.68), up to the calculation of κ) was provided by the results of Reynolds & Potter (1967) for the combined Poiseuille/Couette flow of a homogeneous fluid. The results of their calculations for the value of 0.3 for their parameter u_w were transformed into the non-dimensionalization used here and are presented, with our calculations for the same profile, in table A5. (Note that $U_p \equiv 1/(1+u_w)$ is the relation between the parameters describing the basic flow profiles. Also, the Reynolds number in our scaling is given by $(1+u_w)$ times the Reynolds number in the scaling of Reynolds & Potter (1967).) Again we have good agreement between the two sets of results, and hence we can be reasonably confident that our program to calculate κ is correct.

Table A5. Constants for nonlinear analysis of combined Poiseuille–Couette flow of a homogeneous fluid

	from Reynolds & Potter (1967) (for $u_w = 0.3$)	numerical solution (for $U_p = 0.7690$) with step length of 1/100
R_c	14040	14059
k_c	0.486	0.479
c_r	0.3215	0.3211
κ	(52.8, −106)	(51.0, −105)

Appendix B

In this appendix we give the details of the systems of equations leading to the definitions of the critical constants a_1, d_1 and a_2.

In the matrix formulation of the linear stability problem (§§2.3.1 and 2.3.2) the Orr–Sommerfeld equation at critical conditions was written as

$$\left.\begin{array}{c} W_1' - S_1 W_1 = 0 \\ \text{no slip on boundaries,} \end{array}\right\} \qquad (2.40)$$

where W_1 and S_1 are terms in the expansion (2.39) and

$$S_1 = \begin{bmatrix} \sigma_r'/\sigma_r & -ik_c & 0 & 0 \\ (k_c^2 + \sigma_r''/\sigma_r)/ik_c & -\sigma_r'/\sigma_r & \mu R_c & 0 \\ \dfrac{4ik_c\,\sigma_r'}{\sigma_r \mu R_c} + \dfrac{F + k_c^2 T}{\sigma_r \rho R^2} & \dfrac{4k_c^2}{\mu R_c} + \dfrac{ik_c\sigma_r}{\rho} & 0 & ik_c \\ \dfrac{-ik_c\sigma_r}{\rho} & \dfrac{F + k_c^2 T}{\sigma_r \rho R_c^2} & ik_c & 0 \end{bmatrix}. \qquad (B\,1)$$

The quantities T, ρ and μ are the step functions given by (2.31 *b*, *c*, *d*), and σ_r is given by (2.31 *a*) with c replaced by c_r, as at critical conditions c is real.

The system used to define a_1 is

$$\left.\begin{aligned} W'_{10} - S_1 W_{10} &= S_{10} W_1 \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.42}$$

where

$$S_{10} = \begin{bmatrix} -\sigma'_r(a_1+c_r)/k_c\sigma_r^2 & -i & 0 & 0 \\ \dfrac{i\sigma''_r(\sigma_r+c_r+a_1)}{k_c^2\sigma_r^2} - i & \dfrac{\sigma'_r(a_1+c_r)}{k_c\sigma_r^2} & 0 & 0 \\ \left[\dfrac{4i\sigma'_r(\sigma_r-a_1-c_r)}{\mu R_c\sigma_r^2} - \dfrac{F(a_1+c_r)-k_c^2T(2\sigma_r-c_r-a_1)}{\rho k_c\sigma_r^2}\right] & \dfrac{8k_c}{\mu R_c} + \dfrac{i(\sigma_r+a_1+c_r)}{\rho} & 0 & i \\ -\dfrac{i(\sigma_r+a_1+c_r)}{\rho} & -\dfrac{F(a_1+c_r)-k_c^2T(2\sigma_r-c_r-a_1)}{\rho k_c\sigma_r^2} & i & 0 \end{bmatrix}. \tag{B 2}$$

As stated in §2.3.2, S_{10} can be written as $(A+a_1B)$, where A and B are 4×4 matrices independent of a_1, so that the solvability condition (2.43) gives a linear equation for a_1.

Similarly, the system used to define d_1 is

$$\left.\begin{aligned} W'_{11} - S_1 W_{11} &= S_{11} W_1 \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.46}$$

where

$$S_{11} = \begin{bmatrix} i\sigma'_r d_1/k_c\sigma_r^2 & 0 & 0 & 0 \\ \sigma''_r d_1/k_c^2\sigma_r^2 & -i\sigma'_r d_1/k_c\sigma_r^2 & \mu & 0 \\ \left[-\dfrac{4\sigma'_r}{\mu R_c\sigma_r}\left(\dfrac{ik_c}{R_c}+\dfrac{d_1}{\sigma_r}\right) + \dfrac{F+k_c^2T}{\rho\sigma_r R_c^3}\left(\dfrac{id_1}{k_c\sigma_r}-\dfrac{2}{R_c}\right)\right] & -\dfrac{4k_c^2}{\mu R_c^3}+\dfrac{d_1}{\rho} & 0 & 0 \\ -\dfrac{d_1}{\rho} & \dfrac{F+k_c^2T}{\rho\sigma_r R_c^2}\left(\dfrac{id_1}{k_c\sigma_r}-\dfrac{2}{R_c}\right) & 0 & 0 \end{bmatrix}. \tag{B 3}$$

Finally, the system used to define a_2 is

$$\left.\begin{aligned} W'_{12} - S_1 W_{12} &= S_{10} W_{10} + S_{12} W_1 \\ \text{no slip on boundaries,} \end{aligned}\right\} \tag{2.46a}$$

where

$$S_{12} = \begin{bmatrix} s_{11} & 0 & 0 & 0 \\ s_{21} & -s_{11} & 0 & 0 \\ s_{31}+s_{42} & s_{32} & 0 & 0 \\ s_{41} & s_{42} & 0 & 0 \end{bmatrix}. \tag{B 4a}$$

The elements of the matrix in (B 4a) are given by

$$s_{11} = -\frac{i\sigma'_r}{k_c\sigma_r^2}\left[a_2 + \frac{i(\sigma_r+a_1+c_r)\,(a_1+c_r)}{k_c\sigma_r}\right], \quad s_{21} = -\frac{\sigma''}{k_c^2\sigma_r^2}\left[a_2+\frac{i(\sigma_r+a_1+c_r)^2}{k_c\sigma_r}\right], \tag{B 4b, c}$$

$$s_{31} = \frac{4\sigma'_r}{\mu R_c\sigma_r^2}\left[a_2+\frac{i(a_1+c_r)^2}{k_c\sigma_r}\right], \quad s_{41} = \frac{a_2}{\rho}, \quad s_{32} = \frac{4}{\mu R_c} - \frac{a_2}{\rho} \tag{B 4d, e, f}$$

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and
$$s_{42} = -\frac{\mathrm{i}F}{\rho R_c^2 k_c \sigma_r^2}\left[a_2 + \frac{\mathrm{i}(\sigma_r + a_1 + c_r)\,(a_1 + c_r)}{k_c \sigma_r}\right] + \frac{\mathrm{i}T}{\rho R_c^2 \sigma_r^2}\left[-a_2 k_c + 3\mathrm{i}(a_1 + c_r) - \mathrm{i}(\sigma_r + a_1 + c_r)^2/\sigma_r\right]. \tag{B 4g}$$

The differential systems defining the critical constants can also be cast as non-homogeneous fourth-order problems, in which case the adjoint system is (2.35) and the condition (2.36) is used when applying solvability conditions. Thus, by using the expansions (2.38), the fourth-order system leading to the definition of a_1 is

$$\left.\begin{aligned}
&\Phi_{10} = \Phi'_{10} = 0 \quad \text{on} \quad y = 1,\\
&L(k_c, \Phi_{10}) = \mathrm{i}\Phi_1[U''_b + 2k_c^2(U_b - c_r)] - [4k_c/R_c + \mathrm{i}(U_b + a_1)]\,(\Phi''_1 - k_c^2\Phi_1),\\
&\left.\begin{aligned}
&\phi_{10} = \Phi_{10},\\
&\phi'_{10} - \frac{u'_b\phi_{10}}{\sigma_r} + \frac{u'_b(a_1 + c_r)}{k_c\sigma_r^2}\,\phi_1 = (\text{l.h.s.}:\text{l.c.} \to \text{u.c.}),\\
&\phi''_{10} + k_c^2\phi_{10} + 2k_c\phi_1 = \mu(\text{l.h.s.}:\text{l.c.} \to \text{u.c.}),\\
&q(k_c, \phi_{10}) - \frac{6k_c}{\nu R_c}\phi'_1 - \mathrm{i}[\phi'_1(u_b + a_1) - \phi_1 u'_b]\\
&\quad - \mathrm{i}\phi_1\frac{F(-u_b + 2c_r + a_1) - k_c^2 T(3u_b - 4c_r - a_1)}{R_c^2\sigma_r^2}\\
&\quad = \rho(\text{l.h.s.}:\text{l.c.} \to \text{u.c.};\ \nu \to 1;\ T \to 0),
\end{aligned}\right] \quad \text{on} \quad y = 0\\
&l(k_c, \phi_{10}) = \mathrm{i}\phi_1[u''_b + 2k_c^2(u_b - c_r)] - \left[\frac{4k_c}{\nu R_c} + \mathrm{i}(u_b + a_1)\right](\phi''_1 - k_c^2\phi_1)\\
&\phi_{10} = \phi'_{10} = 0 \quad \text{on} \quad y = -d^{-1}.
\end{aligned}\right\} \tag{B 5}$$

The parameters ρ, μ, ν and T are now the constants defined in §2.1 and the shorthand notation introduced in §2.3.3, (2.63), has been used to simplify the interface conditions. The operators L, l, q and Q are defined in (2.61).

Similarly, the system obtained at $O(R - R_c)$ which leads to the definition of d_1 is

$$\left.\begin{aligned}
&\Phi_{11} = \Phi'_{11} = 0 \quad \text{on} \quad y = 1\\
&L(k_c, \Phi_{11}) = -d_1(\Phi''_1 - k_c^2\Phi_1) - (\Phi_1^{(\mathrm{iv})} - 2k_c^2\Phi''_1 + k_c^4\Phi_1)/R_c^2\\
&\left.\begin{aligned}
&\phi_{11} = \Phi_{11}\\
&\phi'_{11} - \frac{u'_b\phi_{11}}{\sigma_r} - \frac{\mathrm{i}u'_b d_1}{k_c\sigma_r^2}\,\phi_1 = (\text{l.h.s.}:\text{l.c.} \to \text{u.c.})\\
&\phi''_{11} + k_c^2\phi_{11} = \mu(\text{l.h.s.}:\text{l.c.} \to \text{u.c.})\\
&q(k_c, \phi_{11}) - d_1\phi'_1 - \mathrm{i}k_c(\phi'_1\sigma_r - \phi_1 u'_b)/R_c - \phi_1\frac{F + k_c^2 T}{\sigma_r R_c^2}\left(\frac{d_1}{\sigma_r} + \frac{\mathrm{i}k_c}{R_c}\right)\\
&\quad = \rho(\text{l.h.s.}:\text{l.c.} \to \text{u.c.};\ T \to 0)
\end{aligned}\right] \quad \text{on} \quad y = 0\\
&l(k_c, \phi_{11}) = -d_1(\phi''_1 - k_c^2\phi_1) - (\phi_1^{(\mathrm{iv})} - 2k_c^2\phi''_1 + k_c^4\phi_1)/\nu R_c^2\\
&\phi_{11} = \phi'_{11} = 0 \quad \text{on} \quad y = -d^{-1}.
\end{aligned}\right\} \tag{B 6}$$

Finally, the system that leads to the definition of a_2 is

$$\Phi_{12} = \Phi'_{12} = 0 \quad \text{on} \quad y = 1,$$

$$\begin{aligned} L(k_c, \Phi_{12}) = {} & \mathrm{i}\Phi_{10}[U''_b + 2k_c^2(U_b - c_r)] - [4k_c/R_c + \mathrm{i}(U_b + a_1)]\,(\Phi''_{10} - k_c^2\Phi_{10}) \\ & + a_2(\Phi''_1 - k_c^2\Phi_1) + \mathrm{i}k_c\Phi_1(3U_b - c_r + 2a_1) - (2/R_c)\,(\Phi''_1 - 3k_c^2\Phi_1), \end{aligned}$$

$$\left.\begin{aligned} & \phi_{12} = \Phi_{12}, \\ & \phi'_{12} - \frac{u'_b\phi_{12}}{\sigma_r} + \phi_{10}\,\frac{u'_b(a_1 + c_r)}{k_c\sigma_r^2} + \phi_1\left[\mathrm{i}a_2 - \frac{(u_b + a_1)\,(a_1 + c_r)}{k_c\sigma_r}\right]\frac{u'_b}{k_c\sigma_r^2} \\ & \qquad = (\text{l.h.s.} : \text{l.c.} \rightarrow \text{u.c.}) \\ & \phi''_{12} + k_c^2\phi_{12} + 2k_c\phi_{10} + \phi_1 = \mu(\text{l.h.s.} : \text{l.c.} \rightarrow \text{u.c.}), \\ & q(k_c, \phi_{12}) - \frac{6k_c}{\nu R_c}\,\phi'_{10} - \mathrm{i}[\phi'_{10}(u_b + a_1) - \phi_{10}u'_b] \\ & \qquad + \mathrm{i}\phi_{10}\,\frac{F(u_b - 2c_r - a_1) + k_c^2\,T(3u_b - 4c_r - a_1)}{R_c^2\sigma_r^2} \\ & \qquad - \frac{3}{\nu R_c}\,\phi'_1 + a_2\left(\phi'_1 + \phi_1\,\frac{F + k_c^2\,T}{R_c^2\sigma_r^2}\right) \\ & \qquad + \frac{\mathrm{i}\phi_1}{\sigma_r^2R_c^2k_c}\left[(a_1 + c_r)^2\,\frac{F + k_c^2\,T}{\sigma_r} + k_c^2\,T(3u_b - 5c_r - 2a_1)\right] \\ & \qquad = \rho(\text{l.h.s.} : \text{l.c.} \rightarrow \text{u.c.};\ \nu \rightarrow 1;\ T \rightarrow 0), \end{aligned}\right\} \quad \text{on} \quad y = 0 \qquad \text{(B 7)}$$

$$\begin{aligned} l(k_c, \phi_{12}) = {} & \mathrm{i}\phi_{10}[u''_b + 2k_c^2(u_b - c_r)] - [4k_c/\nu R_c + \mathrm{i}(u_b + a_1)]\,(\phi''_{10} - k_c^2\phi_{10}) \\ & + a_2(\phi''_1 - k_c^2\phi_1) + \mathrm{i}k_c\phi_1(3u_b - c_r + 2a_1) - (2/\nu R_c)\,(\phi''_1 - 3k_c^2\phi_1) \end{aligned}$$

$$\phi_{12} = \phi'_{12} = 0 \quad \text{on} \quad y = -d^{-1}.$$

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